No. 5.

HARMONIC FUNCTIONS.

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EDITORS' PREFACE.

THE volume called Higher Mathematics, the first edition of which was published in 1896, contained eleven chapters by eleven authors, each chapter being independent of the others, but all supposing the reader to have at least a mathematical training equivalent to that given in classical and engineering colleges. The publication of that volume is now discontinued and the chapters are issued in separate form. In these reissues it will generally be found that the monographs are enlarged by additional articles or appendices which either amplify the former presentation or record recent advances. This plan of publication has been arranged in order to meet the demand of teachers and the convenience of classes, but it is also thought that it may prove advantageous to readers in special lines of mathematical literature.

It is the intention of the publishers and editors to add other monographs to the series from time to time, if the call for the same seems to warrant it. Among the topics which are under consideration are those of elliptic functions, the theory of numbers, the group theory, the calculus of variations, and non-Euclidean geometry; possibly also monographs on branches of astronomy, mechanics, and mathematical physics may be included. It is the hope of the editors that this form of publication may tend to promote mathematical study and research over a wider field than that which the former volume has occupied.

December, 1905.

AUTHOR'S PREFACE.

THIS brief sketch of the Harmonic Functions and their use in Mathematical Physics was written as a chapter of Merriman and Woodward's Higher Mathematics. It was intended to give enough in the way of introduction and illustration to serve as a useful part of the equipment of the general mathematical student, and at the same time to point out to one specially interested in the subject the way to carry on his study and reading toward a broad and detailed knowledge of its more difficult portions.

Fourier's Series, Zonal Harmonics, and Bessel's Functions of the order zero are treated at considerable length, with the intention of enabling the reader to use them in actual work in physical problems, and to this end several valuable numerical tables are included in the text.

CAMBRIDGE, MASS., December, 1905.

CONTENTS.

HARMONIC FUNCTIONS.

ART. 1. HISTORY AND DESCRIPTION.

What is known as the Harmonic Analysis owed its origin and development to the study of concrete problems in various branches of Mathematical Physics, which however all involved the treatment of partial differential equations of the same general form.

The use of Trigonometric Series was first suggested by Daniel Bernouilli in 1753 in his researches on the musical vibrations of stretched elastic strings, although Bessel's Functions had been already (1732) employed by him and by Euler in dealing with the vibrations of a heavy string suspended from one end; and Zonal and Spherical Harmonics were introduced by Legendre and Laplace in 1782 in dealing with the attraction of solids of revolution.

The analysis was greatly advanced by Fourier in 1812–1824 in his remarkable work on the Conduction of Heat, and important additions have been made by Lamé (1839) and by a host of modern investigators.

The differential equations treated in the problems which have just been enumerated are

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2} \tag{1}$$

for the transverse vibrations of a musical string;

$$\frac{\partial^2 y}{\partial t^2} = c^2\left(x\frac{\partial^2 y}{\partial x^2} + \frac{\partial y}{\partial x}\right) \tag{2}$$

for small transverse vibrations of a uniform heavy string suspended from one end;

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0, \tag{3}$$

which is Laplace's equation; and

$$\frac{\partial u}{\partial t} = a^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) \tag{4}$$

for the conduction of heat in a homogeneous solid.

Of these Laplace's equation (3), and (4) of which (3) is a special case, are by far the most important, and we shall concern ourselves mainly with them in this chapter. As to their interest to engineers and physicists we quote from an article in The Electrician of Jan. 26, 1894, by Professor John Perry:

"There is a well-known partial differential equation, which is the same in problems on heat-conduction, motion of fluids, the establishment of electrostatic or electromagnetic potential, certain motions of viscous fluid, certain kinds of strain and stress, currents in a conductor, vibrations of elastic solids, vibrations of flexible strings or elastic membranes, and innumerable other phenomena. The equation has always to be solved subject to certain boundary or limiting conditions, sometimes as to space and time, sometimes as to space alone, and we know that if we obtain any solution of a particular problem, then that is the true and only solution. Furthermore, if a solution, say, of a heat-conduction problem is obtained by any person, that answer is at once applicable to analogous problems in all the other departments of physics. Thus, if Lord Kelvin draws for us the lines of flow in a simple vortex, he has drawn for us the lines of magnetic force about a circular current; if Lord Rayleigh calculates for us the resistance of the mouth of an organ-pipe, he has also determined the end effect of a bar of iron which is magnetized; when Mr. Oliver Heaviside shows his match-

less skill and familiarity with Bessel's functions in solving electromagnetic problems, he is solving problems in heat-conductivity or the strains in prismatic shafts. How difficult it is to express exactly the distribution of strain in a twisted square shaft, for example, and yet how easy it is to understand thoroughly when one knows the perfect-fluid analogy! How easy, again, it is to imagine the electric current density everywhere in a conductor when transmitting alternating currents when we know Mr. Heaviside's viscous-fluid analogy, or even the heat-conduction analogy!

"Much has been written about the correlation of the physical sciences; but when we observe how a young man who has worked almost altogether at heat problems suddenly shows himself acquainted with the most difficult investigations in other departments of physics, we may say that the true correlation of the physical sciences lies in the equation of continuity

$$\frac{\partial u}{\partial t} = a^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)."$$

In the Theory of the Potential Function in the Attraction of Gravitation, and in Electrostatics and Electrodynamics,* V in Laplace's equation (3) is the value of the Potential Function, at any external point (x, y, z), due to any distribution of matter or of electricity; in the theory of the Conduction of Heat in a homogeneous solid † V is the temperature at any point in the solid after the stationary temperatures have been established, and in the theory of the irrotational flow of an incompressible fluid ‡ V is the Velocity Potential Function and (3) is known as the equation of continuity.

If we use spherical coördinates, (3) takes the form

$$\frac{1}{r^2}\left[r\frac{\partial^2(rV)}{\partial r^2} + \frac{1}{\sin\theta}\frac{\partial\left(\sin\theta\frac{\partial V}{\partial\theta}\right)}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2}\right] = 0; \qquad (5)$$

* See Peirce's Newtonian Potential Function. Boston.

† See Fourier's Analytic Theory of Heat. London and New York, 1878; or Riemann's Partielle Differentialgleichungen. Brunswick.

‡ See Lamb's Hydrodynamics. London and New York, 1895.

and if we use cylindrical coördinates, the form

$$\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} = 0. \tag{6}$$

In the theory of the Conduction of Heat in a homogeneous solid,* u in equation (4) is the temperature of any point (x, y, z) of the solid at any time t, and a^2 is a constant determined by experiment and depending on the conductivity and the thermal capacity of the solid.

ART. 2. HOMOGENEOUS LINEAR DIFFERENTIAL EQUATIONS.

The general solution of a differential equation is the equation expressing the most general relation between the primitive variables which is consistent with the given differential equation and which does not involve differentials or derivatives. A general solution will always contain arbitrary (i.e., undetermined) constants or arbitrary functions.

A particular solution of a differential equation is a relation between the primitive variables which is consistent with the given differential equation, but which is less general than the general solution, although included in it.

Theoretically, every particular solution can be obtained from the general solution by substituting in the general solution particular values for the arbitrary constants or particular functions for the arbitrary functions; but in practice it is often easy to obtain particular solutions directly from the differential equation when it would be difficult or impossible to obtain the general solution.

(*a*) If a problem requiring for its solution the solving of a differential equation is determinate, there must always be given in addition to the differential equation enough outside conditions for the determination of all the arbitrary constants or arbitrary functions that enter into the general solution of the equation; and in dealing with such a problem, if the differential equation can be readily solved the natural method of pro-

cedure is to obtain its general solution, and then to determine the constants or functions by the aid of the given conditions.

It often happens, however, that the general solution of the differential equation in question cannot be obtained, and then, since the problem, if determinate, will be solved, if by any means a solution of the equation can be found which will also satisfy the given outside conditions, it is worth while to try to get particular solutions and so to combine them as to form a result which shall satisfy the given conditions without ceasing to satisfy the differential equation.

(*b*) A differential equation is linear when it would be of the first degree if the dependent variable and all its derivatives were regarded as algebraic unknown quantities. If it is linear and contains no term which does not involve the dependent variable or one of its derivatives, it is said to be linear and homogeneous.

All the differential equations given in Art. 1 are linear and homogeneous.

(*c*) If a value of the dependent variable has been found which satisfies a given homogeneous, linear, differential equation, the product formed by multiplying this value by any constant will also be a value of the dependent variable which will satisfy the equation.

For if all the terms of the given equation are transposed to the first member, the substitution of the first-named value must reduce that member to zero; substituting the second value is equivalent to multiplying each term of the result of the first substitution by the same constant factor, which therefore may be taken out as a factor of the whole first member. The remaining factor being zero, the product is zero and the equation is satisfied.

(*d*) If several values of the dependent variable have been found each of which satisfies the given differential equation, their sum will satisfy the equation; for if the sum of the values in question is substituted in the equation, each term of the sum

will give rise to a set of terms which must be equal to zero, and therefore the sum of these sets must be zero.

(*e*) It is generally possible to get by some simple device particular solutions of such differential equations as those we have collected in Art. 1. The object of this chapter is to find methods of so combining these particular solutions as to satisfy any given conditions which are consistent with the nature of the problem in question.

This often requires us to be able to develop any given function of the variables which enter into the expression of these conditions in terms of normal forms suited to the problem with which we happen to be dealing, and suggested by the form of particular solution that we are able to obtain for the differential equation.

These normal forms are frequently sines and cosines, but they are often much more complicated functions known as Legendre's Coefficients, or Zonal Harmonics; Laplace's Coefficients, or Spherical Harmonics; Bessel's Functions, or Cylindrical Harmonics; Lamé's Functions, or Ellipsoidal Harmonies; etc.

ART. 3. PROBLEM IN TRIGONOMETRIC SERIES.

As an illustration let us consider the following problem: A large iron plate π centimeters thick is heated throughout to a uniform temperature of 100 degrees centigrade; its faces are then suddenly cooled to the temperature zero and are kept at that temperature for 5 seconds. What will be the temperature of a point in the middle of the plate at the end of that time? Given $a^2 = 0.185$ in C.G.S. units.

Take the origin of coördinates in one face of the plate and the axis of X perpendicular to that face, and let u be the temperature of any point in the plate t seconds after the cooling begins.

We shall suppose the flow of heat to be directly across the plate so that at any given time all points in any plane parallel

to the faces of the plate will have the same temperature. Then u depends upon a single space-coordinate x; $\frac{\partial u}{\partial y} = 0$ and $\frac{\partial u}{\partial z} = 0$, and (4), Art. 1, reduces to

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}. \tag{1}$$

Obviously, $u = 100^\circ$ when $t = 0$, (2)

$u = 0$ when $x = 0$, (3)

and $u = 0$ when $x = \pi$; (4)

and we need to find a solution of (1) which satisfies the conditions (2), (3), and (4).

We shall begin by getting a particular solution of (1), and we shall use a device which always succeeds when the equation is linear and homogeneous and has constant coefficients.

Assume* $u = e^{\beta x + \gamma t}$, where β and γ are constants; substitute in (1) and divide through by $e^{\beta x + \gamma t}$ and we get $\gamma = a^2 \beta^2$; and if this condition is satisfied, $u = e^{\beta x + \gamma t}$ is a solution of (1).

$u = e^{\beta x + a^2 \beta^2 t}$ is then a solution of (1) no matter what the value of β.

We can modify the form of this solution with advantage. Let $\beta = \mu i$,† then $u = e^{-a^2\mu^2 t} e^{\mu x i}$ is a solution of (1), as is also $u = e^{-a^2\mu^2 t} e^{-\mu x i}$.

By (d), Art. 2,

$$u = e^{-a^2\mu^2 t} \frac{(e^{\mu x i} + e^{-\mu x i})}{2} = e^{-a^2\mu^2 t} \cos \mu x \tag{5}$$

is a solution, as is also

$$u = e^{-a^2\mu^2 t} \frac{(e^{\mu x i} - e^{-\mu x i})}{2i} = e^{-a^2\mu^2 t} \sin \mu x; \tag{6}$$

and μ is entirely arbitrary.

* This assumption must be regarded as purely tentative. It must be tested by substituting in the equation, and is justified if it leads to a solution.

† The letter i will be used to represent $\sqrt{-1}$.

By giving different values to μ we get different particular solutions of (1); let us try to so combine them as to satisfy our conditions while continuing to satisfy equation (1).

$u = e^{-a^2\mu^2 t} \sin \mu x$ is zero when $x = 0$ for all values of μ; it is zero when $x = \pi$ if μ is a whole number. If, then, we write u equal to a sum of terms of the form $Ae^{-a^2m^2t} \sin mx$, where m is an integer, we shall have a solution of (1) (see (*d*), Art. 2) which satisfies (3) and (4).

Let this solution be

$$u = A_1 e^{-a^2t} \sin x + A_2 e^{-4a^2t} \sin 2x + A_3 e^{-9a^2t} \sin 3x + \ldots, \tag{7}$$

$A_1, A_2, A_3, \ldots$ being undetermined constants.

When $t = 0$, (7) reduces to

$$u = A_1 \sin x + A_2 \sin 2x + A_3 \sin 3x + \ldots. \tag{8}$$

If now it is possible to develop unity into a series of the form (8) we have only to substitute the coefficients of that series each multiplied by 100 for $A_1, A_2, A_3 \ldots$ in (7) to have a solution satisfying (1) and all the equations of condition (2), (3), and (4).

We shall prove later (see Art. 6) that

$$1 = \frac{4}{\pi}\left[\sin x + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x + \ldots\right]$$

for all values of x between 0 and π. Hence our solution is

$$u = \frac{400}{\pi}\left[e^{-a^2t} \sin x + \frac{1}{3}e^{-9a^2t} \sin 3x + \frac{1}{5}e^{-25a^2t} \sin 5x + \ldots\right] \tag{9}$$

To get the answer of the numerical problem we have only to compute the value of u when $x = \frac{\pi}{2}$ and $t = 5$ seconds. As there is no object in going beyond tenths of a degree, four-place tables will more than suffice, and no term of (9) beyond the first will affect the result. Since $\sin \frac{\pi}{2} = 1$, we have to compute the numerical value of

$$\frac{400}{\pi}e^{-a^2t} \quad \text{where} \quad a^2 = 0.185 \quad \text{and} \quad t = 5.$$

$$\begin{array}{lll}
\log a^2 & = 9.2672 - 10 \\
\log t & = \underline{0.6990} \\
\log a^2t & = 9.9662 - 10 \\
\log \log e & = \underline{9.6378} - 10 \\
\log \log e^{a^2t} & = 9.6040 - 10 \\
\log e^{a^2t} & = \underline{0.4018}
\end{array}
\qquad
\begin{array}{lll}
\log 400 & = 2.6021 \\
\operatorname{colog} \pi & = 9.5059 - 10 \\
\operatorname{colog} e^{a^2t} & = \underline{9.5982} - 10 \\
\log u & = 1.7062 \\
u & = 50^\circ.8.
\end{array}$$

If the breadth of the plate had been c centimeters instead of π centimeters it is easy to see that we should have needed the development of unity in a series of the form

$$A_1 \sin \frac{\pi x}{c} + A_2 \sin \frac{2\pi x}{c} + A_3 \sin \frac{3\pi x}{c} + \ldots .$$

Prob. 1. An iron slab 50 centimeters thick is heated to the temperature 100 degrees Centigrade throughout. The faces are then suddenly cooled to zero degrees, and are kept at that temperature for 10 minutes. Find the temperature of a point in the middle of the slab, and of a point 10 centimeters from a face at the end of that time. Assume that

$$1 = \frac{4}{\pi}\left(\sin \frac{\pi x}{c} + \frac{1}{3} \sin \frac{3\pi x}{c} + \frac{1}{5} \sin \frac{5\pi x}{c} + \ldots\right) \text{ from } x = 0 \text{ to } x = c.$$

Ans. $84^\circ.0$; $49^\circ.4$.

ART. 4. PROBLEM IN ZONAL HARMONICS.

As a second example let us consider the following problem: Two equal thin hemispherical shells of radius unity placed together to form a spherical surface are separated by a thin layer of air. A charge of statical electricity is placed upon one hemisphere and the other hemisphere is connected with the ground, the first hemisphere is then found to be at potential 1, the other hemisphere being of course at potential zero. At what potential is any point in the "field of force" due to the charge?

We shall use spherical coordinates and shall let V be the potential required. Then V must satisfy equation (5), Art. 1.

But since from the symmetry of the problem V is obviously independent of ϕ, if we take the diameter perpendicular to the plane separating the two conductors as our polar axis, $\dfrac{\partial^2 V}{\partial \phi^2}$ is zero, and our equation reduces to

$$\frac{r\partial^2(rV)}{\partial r^2} + \frac{1}{\sin\theta}\frac{\partial\left(\sin\theta\dfrac{\partial V}{\partial\theta}\right)}{\partial\theta} = 0. \tag{1}$$

V is given on the surface of our sphere, hence

$$V = f(\theta) \quad \text{when} \quad r = 1, \tag{2}$$

where $f(\theta) = 1$ if $0 < \theta < \dfrac{\pi}{2}$, and $f(\theta) = 0$ if $\dfrac{\pi}{2} < \theta < \pi$.

Equation (2) and the implied conditions that V is zero at an infinite distance and is nowhere infinite are our conditions.

To find particular solutions of (1) we shall use a method which is generally effective. Assume* that $V = R\Theta$ where R is a function of r but not of θ, and O is a function of θ but not of r. Substitute in (1) and reduce, and we get

$$\frac{1}{R}\frac{rd^2(rR)}{dr^2} = -\frac{1}{\Theta\sin\theta}\frac{d\left(\sin\theta\dfrac{d\Theta}{d\theta}\right)}{d\theta}. \tag{3}$$

Since the first member of (3) does not contain θ and the second does not contain r and the two members are identically equal, each must be equal to a constant. Let us call this constant, which is wholly undetermined, $m(m+1)$; then

$$\frac{r}{R}\frac{d^2(rR)}{dr^2} = -\frac{1}{\Theta\sin\theta}\frac{d\left(\sin\theta\dfrac{d\Theta}{d\theta}\right)}{d\theta} = m(m+1);$$

whence
$$r\frac{d^2(rR)}{dr^2} - m(m+1)R = 0, \tag{4}$$

and
$$\frac{1}{\sin\theta}\frac{d\left(\sin\theta\dfrac{d\Theta}{d\theta}\right)}{d\theta} + m(m+1)\Theta = 0. \tag{5}$$

* See the first foot-note on page 175.

Equation (4) can be expanded into

$$r^2\frac{d^2R}{dr^2} + 2r\frac{dR}{dr} - m(m+1)R = 0,$$

and can be solved by elementary methods. Its complete solution is

$$R = Ar^m + Br^{-m-1}. \tag{6}$$

Equation (5) can be simplified by changing the independent variable to x where $x = \cos\theta$. It becomes

$$\frac{d}{dx}\left[(1 - x^2)\frac{d\Theta}{dx}\right] + m(m+1)\Theta = 0, \tag{7}$$

an equation which has been much studied and which is known as Legendre's Equation.

We shall restrict m, which is wholly undetermined, to positive whole values, and we can then get particular solutions of (7) by the following device:

Assume* that O can be expressed as a sum or a series of terms involving whole powers of x multiplied by constant coefficients.

Let $O = \Sigma a_n x^n$ and substitute in (7). We get

$$\Sigma[n(n-1)a_n x^{n-2} - n(n+1)a_n x^n + m(m+1)a_n x^n] = 0, \tag{8}$$

where the symbol Σ indicates that we are to form all the terms we can by taking successive whole numbers for n.

Since (8) must be true no matter what the value of x, the coefficient of any given power of x, as for instance x^k, must vanish. Hence

$$(k+2)(k+1)a_{k+2} - k(k+1)a_k + m(m+1)a_k = 0,$$

and
$$a_{k+2} = -\frac{m(m+1) - k(k+1)}{(k+1)(k+2)}a_k. \tag{9}$$

If now any set of coefficients satisfying the relation (9) be taken, $\Theta = \Sigma a_k x^k$ will be a solution of (7).

If $k = m$, then $a_{k+2} = 0$, $a_{k+4} = 0$, etc.

* See the first foot-note on page ~~175~~. 17

Since it will answer our purpose if we pick out the simplest set of coefficients that will obey the condition (9), we can take a set including a_m.

Let us rewrite (9) in the form

$$a_k = -\frac{(k+2)(k+1)a_{k+2}}{(m-k)(m+k-1)}. \tag{10}$$

We get from (10), beginning with $k = m - 2$,

$$a_{m-2} = -\frac{m(m-1)}{2.(2m-1)}a_m,$$

$$a_{m-4} = \frac{m(m-1)(m-2)(m-3)}{2.4.(2m-1)(2m-3)}a_m,$$

$$a_{m-6} = -\frac{m(m-1)(m-2)(m-3)(m-4)(m-5)}{2.4.6.(2m-1)(2m-3)(2m-5)}a_m, \text{ etc.}$$

If m is even we see that the set will end with a_0; if m is odd, with a_1.

$$\Theta = a_m\Big[x^m - \frac{m(m-1)}{2.(2m-1)}x^{m-2} + \frac{m(m-1)(m-2)(m-3)}{2.4.(2m-1)(2m-3)}x^{m-} - \ldots\Big],$$

where a_m is entirely arbitrary, is, then, a solution of (7). It is found convenient to take a_m equal to

$$\frac{(2m-1)(2m-3)\ldots 1}{m!},$$

and it will be shown later that with this value of a_m, O $=1$ when $x = 1$.

O is a function of x and contains no higher powers of x than x^m. It is usual to write it as $P_m(x)$.

We proceed to write out a few values of $P_m(x)$ from the formula

$$P_m(x) = \frac{(2m-1)(2m-3)\ldots 1}{m!}\Big[x^m - \frac{m(m-1)}{2.(2m-1)}x^{m-2} + \frac{m(m-1)(m-2)(m-3)}{2.4.(2m-1)(2m-3)}x^{m-4} - \ldots\Big] \tag{11}$$

We have:

$$\left.\begin{aligned}
&P_0(x) = 1 \quad\text{or}\quad P_0(\cos\theta) = 1,\\
&P_1(x) = x \quad\text{or}\quad P_1(\cos\theta) = \cos\theta,\\
&P_2(x) = \tfrac{1}{2}(3x^2 - 1) \quad\text{or}\quad P_2(\cos\theta) = \tfrac{1}{2}(3\cos^2\theta - 1),\\
&P_3(x) = \tfrac{1}{2}(5x^3 - 3x) \quad\text{or}\quad P_3(\cos\theta) = \tfrac{1}{2}(5\cos^3\theta - 3\cos\theta),\\
&P_4(x) = \tfrac{1}{8}(35x^4 - 30x^2 + 3) \quad\text{or}\\
&\qquad P_4(\cos\theta) = \tfrac{1}{8}(35\cos^4\theta - 30\cos^2\theta + 3),\\
&P_5(x) = \tfrac{1}{8}(63x^5 - 70x^3 + 15x) \quad\text{or}\\
&\qquad P_5(\cos\theta) = \tfrac{1}{8}(63\cos^5\theta - 70\cos^3\theta + 15\cos\theta).
\end{aligned}\right\} \quad (12)$$

We have obtained $O = P_m(x)$ as a particular solution of (7), and $O = P_m(\cos\theta)$ as a particular solution of (5). $P_m(x)$ or $P_m(\cos\theta)$ is a new function, known as a Legendre's Coefficient, or as a Surface Zonal Harmonic, and occurs as a normal form in many important problems.

$V = r^m P_m(\cos\theta)$ is a particular solution of (1), and $r^m P_m(\cos\theta)$ is sometimes called a Solid Zonal Harmonic.

$$V = A_0P_0(\cos\theta) + A_1rP_1(\cos\theta) + A_2r^2P_2(\cos\theta) + A_3r^3P_3(\cos\theta) + \ldots \quad (13)$$

satisfies (1), is not infinite at any point within the sphere, and reduces to

$$V = A_0P_0(\cos\theta) + A_1P_1(\cos\theta) + A_2P_2(\cos\theta) + A_3P_3(\cos\theta) + \ldots \quad (14)$$

when $r = 1$.

$$V = \frac{A_0P_0(\cos\theta)}{r} + \frac{A_1P_1(\cos\theta)}{r^2} + \frac{A_2P_2(\cos\theta)}{r^3} + \frac{A_3P_3(\cos\theta)}{r^4} + \ldots \quad (15)$$

satisfies (1), is not infinite at any point without the sphere, is equal to zero when $r = \infty$, and reduces to (14) when $r = 1$.

If then we can develop $f(\theta)$ [see eq. (2)] into a series of the form (14), we have only to put the coefficients of this series in place of the $A_0, A_1, A_2, \ldots$ in (13) to get the value of V for a point within the sphere, and in (15) to get the value of V at a point without the sphere.

We shall see later (Art. 16, Prob. 22) that if $f(\theta) = 1$ for $0 < \theta < \frac{\pi}{2}$ and $f(\theta) = 0$ for $\frac{\pi}{2} < \theta < \pi$,

$$f(\theta) = \frac{1}{2} + \frac{3}{4} P_1(\cos\theta) - \frac{7}{8} \cdot \frac{1}{2} \cdot P_3(\cos\theta)$$
$$+ \frac{11}{12} \cdot \frac{1 \cdot 3}{2 \cdot 4} P_5(\cos\theta) - \ldots \quad (16)$$

Hence our required solution is

$$V = \frac{1}{2} + \frac{3}{4} r P_1(\cos\theta) - \frac{7}{8} \cdot \frac{1}{2} \cdot r^3 P_3(\cos\theta)$$
$$+ \frac{11}{12} \cdot \frac{1 \cdot 3}{2 \cdot 4} r^5 P_5(\cos\theta) - \ldots \quad (17)$$

at an internal point; and

$$V = \frac{1}{2r} + \frac{3}{4} \frac{1}{r^2} P_1(\cos\theta) - \frac{7}{8} \cdot \frac{1}{2} \frac{1}{r^4} P_3(\cos\theta)$$
$$+ \frac{11}{12} \cdot \frac{1 \cdot 3}{2 \cdot 4} \frac{1}{r^6} P_5(\cos\theta) - \ldots \quad (18)$$

at an external point.

If $r = \frac{1}{4}$ and $\theta = 0$, (17) reduces to

$$V = \frac{1}{2} + \frac{3}{4} \cdot \frac{1}{4} - \frac{7}{8} \cdot \frac{1}{2} \cdot \frac{1}{4^3} + \frac{11}{12} \cdot \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{1}{4^5} \ldots, \text{ since } P_m(1) = 1.$$

To two decimal places $V = 0.68$, and the point $r = \frac{1}{4}$, $\theta = 0$ is at potential 0.68.

If $r = 5$ and $\theta = \frac{\pi}{4}$, (18) and Table I, at the end of this chapter, give

$$V = \frac{1}{2 \cdot 5} + \frac{3}{4} \cdot \frac{1}{5^2} \cdot 0.7071 + \frac{7}{8} \cdot \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{1}{5^4} \cdot 0.1768 + \ldots = 0.12,$$

and the point $r = 5$, $\theta = \frac{\pi}{4}$ is at potential 0.12.

If the radius of the conductor is a instead of unity, we have only to replace r by $\frac{r}{a}$ in (17) and (18).

Prob. 2. One half the surface of a solid sphere 12 inches in diameter is kept at the temperature zero and the other half at 100 degrees centigrade until there is no longer any change of temperature at any point within the sphere. Required the temperature of the center; of any point in the diametral plane separating the hot and cold hemispheres; of points 2 inches from the center and in the axis of symmetry; and of points 3 inches from the center in a diameter inclined at an angle of 45° to the axis of symmetry.

Ans. 50°; 50°; 73°.9; 26°.1; 77°.1; 22°.9.

$r = 0$ $\theta = 90°$

ART. 5. PROBLEM IN BESSEL'S FUNCTIONS.

As a last example we shall take the following problem: The base and convex surface of a cylinder 2 feet in diameter and 2 feet high are kept at the temperature zero, and the upper base at 100 degrees centigrade. Find the temperature of a point in the axis one foot from the base, and of a point 6 inches from the axis and one foot from the base, after the permanent state of temperatures has been set up.

If we use cylindrical coördinates and take the origin in the base we shall have to solve equation (6), Art. 1; or, representing the temperature by u and observing that from the symmetry of the problem u is independent of ϕ,

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} = 0, \tag{1}$$

subject to the conditions

$$u = 0 \quad \text{when} \quad z = 0, \tag{2}$$

$$u = 0 \quad \text{"} \quad r = 1, \tag{3}$$

$$u = 100 \quad \text{"} \quad z = 2. \tag{4}$$

Assume $u = RZ$ where R is a function of r only and Z of z only; substitute in (1) and reduce.

We get
$$\frac{1}{R}\frac{d^2R}{dr^2} + \frac{1}{rR}\frac{dR}{dr} = -\frac{1}{Z}\frac{d^2Z}{dz^2}. \tag{5}$$

The first member of (5) does not contain z; therefore the second member cannot. The second member of (5) does not

contain r; therefore the first member cannot. Hence each member of (5) is a constant, and we can write (5)

$$\frac{1}{R}\frac{d^2R}{dr^2} + \frac{1}{rR}\frac{dR}{dr} = -\frac{1}{Z}\frac{d^2Z}{dz^2} = -\mu^2, \tag{6}$$

when μ^2 is entirely undetermined.

Hence
$$\frac{d^2Z}{dz^2} - \mu^2 Z = 0, \tag{7}$$

and
$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + \mu^2 R = 0. \tag{8}$$

Equation (7) is easily solved, and its general solution is

$$Z = Ae^{\mu z} + Be^{-\mu z},$$ or the equivalent form
$$Z = C\cosh(\mu z) + D\sinh(\mu z). \tag{9}$$

We can reduce (8) slightly by letting $\mu r = x$, and it becomes

$$\frac{d^2R}{dx^2} + \frac{1}{x}\frac{dR}{dx} + R = 0. \tag{10}$$

Assume, as in Art. 4, that R can be expressed in terms of whole powers of x. Let $R = \Sigma a_n x^n$ and substitute in (10). We get

$$\Sigma[n(n-1)a_n x^{n-2} + na_n x^{n-2} + a_n x^n] = 0,$$

an equation which must be true, no matter what the value of x. The coefficient of any given power of x, as x^{k-2}, must, then, vanish, and

$$k(k-1)a_k + ka_k + a_{k-2} = 0,$$

or
$$k^2 a_k + a_{k-2} = 0,$$

whence we obtain
$$a_{k-2} = -k^2 a_k \tag{11}$$

as the only relation that need be satisfied by the coefficients in order that $R = \Sigma a_k x^k$ shall be a solution of (10).

If $\qquad k = 0, \quad a_{k-2} = 0, \quad a_{k-4} = 0, \quad$ etc.

We can, then, begin with $k = 0$ as the lowest subscript.

From (11) $$a_k = -\frac{a_{k-2}}{k^2}.$$

Then $a_2 = -\frac{a_0}{2^2}$, $a_4 = \frac{a_0}{2^2 . 4^2}$, $a_6 = -\frac{a_0}{2^2 . 4^2 . 6^2}$, etc.

Hence $$R = a_0\left[1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 . 4^2} - \frac{x^6}{2^2 . 4^2 . 6^2} + \ldots\right],$$

where a_0 may be taken at pleasure, is a solution of (10), provided the series is convergent.

Take $a_0 = 1$, and then $R = J_0(x)$ where

$$J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 . 4^2} - \frac{x^6}{2^2 . 4^2 . 6^2} + \frac{x^8}{2^2 . 4^2 . 6^2 . 8^2} - \ldots \quad (12)$$

is a solution of (10).

$J_0(x)$ is easily shown to be convergent for all values real or imaginary of x, it is a new and important form, and is called a Bessel's Function of the zero order, or a Cylindrical Harmonic.

Equation (10) was obtained from (8) by the substitution of $x = \mu r$; therefore

$$R = J_0(\mu r) = 1 - \frac{(\mu r)^2}{2^2} + \frac{(\mu r)^4}{2^2 . 4^2} - \frac{(\mu r)^6}{2^2 . 4^2 . 6^2} + \ldots$$

is a solution of (8), no matter what the value of μ; and $u = J_0(\mu r) \sinh(\mu z)$ and $u = J_0(\mu r) \cosh(\mu z)$ are solutions of (1). $u = J_0(\mu r) \sinh(\mu z)$ satisfies condition (2) whatever the value of μ. In order that it should satisfy condition (3) μ must be so taken that

$$J_0(\mu) = 0; \qquad J_0(\mu a) = 0 \quad (13)$$

that is, μ must be a root of the transcendental equation (13).

It was shown by Fourier that $J_0(\mu) = 0$ has an infinite number of real positive roots, any one of which can be obtained to any required degree of approximation without serious difficulty. Let $\mu_1, \mu_2, \mu_3, \ldots$ be these roots; then

$$u = A_1 J_0(\mu_1 r) \sinh(\mu_1 z) + A_2 J_0(\mu_2 r) \sinh(\mu_2 z) + A_3 J_0(\mu_3 r) \sinh(\mu_3 z) + \ldots \quad (14)$$

is a solution of (1) which satisfies (2) and (3).

If now we can develop unity into a series of the form

$$1 = B_1 J_0(\mu_1 r) + B_2 J_0(\mu_2 r) + B_3 J_0(\mu_3 r) + \ldots,$$

$$u = 100\left[\frac{B_1 \sinh(\mu_1 z)}{\sinh(2\mu_1)} J_0(\mu_1 r) + \frac{B_2 \sinh(\mu_2 z)}{\sinh(2\mu_2)} J_0(\mu_2 r) + \ldots\right] \quad (15)$$

satisfies (1) and the conditions (2), (3), and (4).

We shall see later (Art. 21) that if $J_1(x) = -\dfrac{dJ_0(x)}{dx}$

$$1 = 2\left[\frac{J_0(\mu_1 r)}{\mu_1 J_1(\mu_1)} + \frac{J_0(\mu_2 r)}{\mu_2 J_1(\mu_2)} + \frac{J_0(\mu_3 r)}{\mu_3 J_1(\mu_3)} + \ldots\right] \quad (16)$$

for values of $r < 1$.

Hence

$$u = 200\left[\frac{J_0(\mu_1 r)\sinh(\mu_1 z)}{\mu_1 J_1(\mu_1)\sinh(2\mu_1)} + \frac{J_0(\mu_2 r)\sinh(\mu_2 z)}{\mu_2 J_1(\mu_2)\sinh(2\mu_2)} + \ldots\right] \quad (17)$$

is our required solution.

At the point $r = 0$, $z = 1$ (17) reduces to

$$u = 200\left[\frac{\sinh\mu_1}{\mu_1 J_1(\mu_1)\sinh(2\mu_1)} + \frac{\sinh\mu_2}{\mu_2 J_1(\mu_2)\sinh(2\mu_2)} + \ldots\right]$$

$$= 100\left[\frac{1}{\mu_1 J_1(\mu_1)\cosh\mu_1} + \frac{1}{\mu_2 J_1(\mu_2)\cosh\mu_2} + \ldots\right],$$

since $J_0(0) = 1$ and $\sinh(2x) = 2\sinh x\cosh x$.

If we use a table of Hyperbolic functions* and Tables II and III, at the end of this chapter, the computation of the value of u is easy. We have

$\mu_1 =$	2.405	$\mu_2 =$	5.520
$J_1(\mu_1) =$	0.5190	$J_1(\mu_2) =$	-0.3402
colog $\mu_1 =$	$9.6189 - 10$	colog $\mu_2 =$	$9.2581 - 10$
" $J_1(\mu_1) =$	0.2848	" $J_1(\mu_2) =$	$0.4683n$
" $\cosh\mu_1 =$	$9.2530 - 10$	" $\cosh\mu_2 =$	$7.9037 - 10$
	$9.1567 - 10$		$7.6301n - 10$

* See Chapter IV, pp. 162, 163, for a four-place table on hyperbolic functions.

$$(\mu_1 J_1(\mu_1)\cosh\mu_1)^{-1} = 0.1434$$
$$(\mu_2 J_1(\mu_2)\cosh\mu_2)^{-1} = -0.0058$$
$$\overline{0.1376}; \qquad u = 13^\circ.8$$

At the point $r = \frac{1}{2}$, $z = 1$, (17), reduces to

$$u = 100\left[\frac{J_0(\frac{1}{2}\mu_1)}{\mu_1 J_1(\mu_1)\cosh\mu_1} + \frac{J_0(\frac{1}{2}\mu_2)}{\mu_2 J_1(\mu_2)\cosh\mu_2} + \ldots\right].$$

$$J_0(\tfrac{1}{2}\mu_1) = 0.6698$$

$$\log J_0(\tfrac{1}{2}\mu_1) = 9.8259 - 10$$
$$\text{colog}\ \mu_1 J_1(\mu_1)\cosh\mu_1 = 9.1567 - 10$$
$$\overline{8.9826} - 10;$$

$$J_0(\tfrac{1}{2}\mu_2) = -0.1678$$

$$\log J_0(\tfrac{1}{2}\mu_2) = 9.2248n - 10$$
$$\text{colog}\ \mu_2 J_1(\mu_2)\cosh\mu_2 = 7.6301n - 10$$
$$\overline{6.8549} - 10;$$

$$\frac{J_0(\frac{1}{2}\mu_1)}{\mu_1 J_1(\mu_1)\cosh\mu_1} = 0.0961$$

$$\frac{J_0(\frac{1}{2}\mu_2)}{\mu_2 J_1(\mu_2)\cosh\mu_2} = \frac{0.0007}{0.0968}; \qquad u = 9^\circ.7$$

If the radius of the cylinder is a and the altitude b, we have only to replace μ by μa in (13); $2\mu_1$, $2\mu_2$, ... in the denominators of (15) and (17) by $\mu_1 b$, $\mu_2 b$, ...; and μ_1, μ_2, μ_3, ... in the denominators of (16) and (17) by $\mu_1 a$, $\mu_2 a$, $\mu_3 a$, ...

Prob. 3. One base and the convex surface of a cylinder 20 centimeters in diameter and 30 centimeters high are kept at zero temperature and the other base at 100 degrees Centigrade. Find the temperature of a point in the axis and 20 centimeters from the cold base, and of a point 5 centimeters from the axis and 20 centimeters from the cold base after the temperatures have ceased to change.

Ans. $13^\circ.9$; $9^\circ.6$.

ART. 6. THE SINE SERIES.

As we have seen in Art. 3, it is sometimes important to be able to express a given function of a variable, x, in terms of sines of multiples of x. The problem in its general form was first solved by Fourier in his "Théorie Analytique de la Chaleur" (1822), and its solution plays an important part in most branches of Mathematical Physics.

Let us endeavor to so develop a given function of x, $f(x)$, in terms of $\sin x$, $\sin 2x$, $\sin 3x$, etc., that the function and the series shall be equal for all values of x between 0 and π.

We can of course determine the coefficients $a_1, a_2, a_3, \ldots a_n$ so that the equation

$$f(x) = a_1 \sin x + a_2 \sin 2x + a_3 \sin 3x + \ldots + a_n \sin nx \quad (1)$$

shall hold good for any n arbitrarily chosen values of x between 0 and π; for we have only to substitute those values in turn in (1) to get n equations of the first degree, in which the n coefficients are the only unknown quantities.

For instance, we can take the n equidistant values Δx, $2\Delta x$, $3\Delta x, \ldots n\Delta x$, where $\Delta x = \dfrac{\pi}{n+1}$, and substitute them for x in (1). We get

$$\left.\begin{aligned}
f(\Delta x) &= a_1 \sin \Delta x + a_2 \sin 2\Delta x + a_3 \sin 3\Delta x + \ldots \\
&\qquad + a_n \sin n\Delta x, \\
f(2\Delta x) &= a_1 \sin 2\Delta x + a_2 \sin 4\Delta x + a_3 \sin 6\Delta x + \ldots \\
&\qquad + a_n \sin 2n\Delta x, \\
f(3\Delta x) &= a_1 \sin 3\Delta x + a_2 \sin 6\Delta x + a_3 \sin 9\Delta x + \ldots \\
&\qquad + a_n \sin 3n\Delta x, \\
&\vdots \\
f(n\Delta x) &= a_1 \sin n\Delta x + a_2 \sin 2n\Delta x + a_3 \sin 3n\Delta x + \ldots \\
&\qquad + a_n \sin n^2\Delta x,
\end{aligned}\right\} \quad (2)$$

n equations of the first degree, to determine the n coefficients $a_1, a_2, a_3, \ldots a_n$.

Not only can equations (2) be solved in theory, but they can be actually solved in any given case by a very simple and

ingenious method due to Lagrange,* and any coefficient a_m can be expressed in the form

$$a_m = \frac{2}{n+1}\sum_{\kappa=1}^{\kappa=n} f(\kappa\Delta x)\sin(\kappa m\Delta x). \qquad (3)$$

If now n is indefinitely increased the values of x for which (1) holds good will come nearer and nearer to forming a continuous set; and the limiting value approached by a_m will probably be the corresponding coefficient in the series required to represent $f(x)$ for all values of x between zero and π.

Remembering that $(n+1)\Delta x = \pi$, the limiting value in question is easily seen to be

$$a_m = \frac{2}{\pi}\int_0^\pi f(x)\sin mx dx. \qquad (4)$$

This value can be obtained from equations (2) by the following device without first solving the equations:

Let us multiply each equation in (2) by the product of Δx and the coefficient of a_m in the equation in question, add the equations, and find the limiting form of the resulting equation as n increases indefinitely.

The coefficient of any a, a_κ in the resulting equation is

$$\sin\kappa\Delta x \sin m\Delta x\,.\,\Delta x + \sin 2\kappa\Delta x \sin 2m\Delta x\,.\,\Delta x + \ldots \\ + \sin n\kappa\Delta x \sin nm\Delta x\,.\,\Delta x.$$

Its limiting value, since $(n+1)\Delta x = \pi$, is

$$\int_0^\pi \sin\kappa x \sin mx\,.\,dx;$$

but

$$\int_0^\pi \sin\kappa x \sin mx\,.\,dx = \tfrac{1}{2}\int_0^\pi [\cos(m-\kappa)x - \cos(m+\kappa)x]dx = 0$$

if m and κ are not equal.

* See Riemann's Partielle Differentialgleichungen, or Byerly's Fourier's Series and Spherical Harmonics.

The coefficient of a_m is

$$\Delta x(\sin^2 m\Delta x + \sin^2 2m\Delta x + \sin^2 3m\Delta x + \ldots + \sin^2 nm\Delta x).$$

Its limiting value is

$$\int_0^\pi \sin^2 mx\,.\,dx = \frac{\pi}{2}.$$

The first member is

$$f(\Delta x)\sin m\Delta x\,.\,\Delta x + f(2\Delta x)\sin 2m\Delta x\,.\,\Delta x + \ldots$$
$$+ f(n\Delta x)\sin mn\Delta x\,.\,\Delta x,$$

and its limiting value is

$$\int_0^\pi f(x)\sin mx\,.\,dx.$$

Hence the limiting form approached by the final equation as n is increased is

$$\int_0^\pi f(x)\sin mx\,.\,dx = \frac{\pi}{2}a_m.$$

Whence $$a_m = \frac{2}{\pi}\int_0^\pi f(x)\sin mx\,.\,dx \qquad (5)$$

as before.

This method is practically the same as multiplying the equation

$$f(x) = a_1 \sin x + a_2 \sin 2x + a_3 \sin 3x + \ldots \qquad (6)$$

by $\sin mx\,.\,dx$ and integrating both members from zero to π.

It is important to realize that the considerations given in this article are in no sense a demonstration, but merely establish a probability.

An elaborate investigation * into the validity of the development, for which we have not space, entirely confirms the results formulated above, provided that between $x = 0$ and $x = \pi$ the

* See Art. 10 for a discussion of this question.

function is finite and single-valued, and has not an infinite number of discontinuities or of maxima or minima.

It is to be noted that the curve represented by $y=f(x)$ need not follow the same mathematical law throughout its length, but may be made up of portions of entirely different curves. For example, a broken line or a locus consisting of finite parts of several different and disconnected straight lines can be represented perfectly well by $y =$ a sine series.

As an example of the application of formula (5) let us take the development of unity.

Here $$f(x) = 1.$$

$$a_m = \frac{2}{\pi}\int_0^\pi \sin mx \, . \, dx;$$

$$\int \sin mx \, . \, dx = -\frac{\cos mx}{m}.$$

$$\int_0^\pi \sin mx \, . \, dx = \frac{1}{m}(1 - \cos m\pi) = \frac{1}{m}[1 - (-1)^m]$$

$$= 0 \text{ if } m \text{ is even}$$

$$= \frac{2}{m} \text{ if } m \text{ is odd.}$$

Hence $$1 = \frac{4}{\pi}\left(\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \frac{\sin 7x}{7} + \ldots\right). \quad (7)$$

It is to be noticed that (7) gives at once a sine development for any constant c. It is,

$$c = \frac{4c}{\pi}\left(\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \ldots\right). \quad (8)$$

Prob. 4. Show that for values of x between zero and π

(a) $$x = 2\left[\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \ldots\right],$$

(b) $$f(x) = \frac{4}{\pi}\left[\frac{\sin x}{1^2} - \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} - \frac{\sin 7x}{7^2} + \ldots\right]$$

if $f(x) = x$ for $0 < x < \frac{\pi}{2}$, and $f(x) = \pi - x$ for $\frac{\pi}{2} < x < \pi$.

(c) $f(x) =$

$$\frac{2}{\pi}\left[\frac{\sin x}{1} + \frac{2 \sin 2x}{2} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \frac{2 \sin 6x}{6} + \frac{\sin 7x}{7} + \ldots\right]$$

if $f(x) = 1$ for $0 < x < \frac{\pi}{2}$, and $f(x) = 0$ for $\frac{\pi}{2} < x < \pi$.

(d) $\sinh x =$

$$\frac{2 \sinh \pi}{\pi}\left[\frac{1}{2} \sin x - \frac{2}{5} \sin 2x + \frac{3}{10} \sin 3x - \frac{4}{17} \sin 4x + \ldots\right].$$

(e) $x^2 =$

$$\frac{2}{\pi}\left[\left(\frac{\pi^2}{1} - \frac{4}{1^3}\right) \sin x - \frac{\pi^2}{2} \sin 2x + \left(\frac{\pi^2}{3} - \frac{4}{3^3}\right)\sin 3x - \frac{\pi^2}{4}\sin 4x + \ldots\right].$$

Art. 7. The Cosine Series.

Let us now try to develop a given function of x in a series of cosines, using the method suggested by the last article.

Assume

$$f(x) = b_0 + b_1 \cos x + b_2 \cos 2x + b_3 \cos 3x + \ldots \qquad (1)$$

To determine any coefficient b_m multiply (1) by $\cos mx \, . \, dx$ and integrate each term from 0 to π.

$$\int_0^\pi b_0 \cos mx \, . \, dx = 0.$$

$$\int_0^\pi b_k \cos kx \cos mx \, . \, dx = 0, \quad \text{if } m \text{ and } k \text{ are not equal.}$$

$$\int_0^\pi b_m \cos^2 mx \;\; dx = \frac{\pi}{2} b_m, \quad \text{if } m \text{ is not zero.}$$

$$\text{Hence } b_m = \frac{2}{\pi}\int_0^\pi f(x) \cos mx \, . \, dx, \qquad (2)$$

if m is not zero.

To get b_0 multiply (1) by dx and integrate from zero to π.

$$\int_0^\pi b_0 dx = b_0 \pi,$$

$$\int_0^\pi b_k \cos kx \, . \, dx = 0.$$

Hence
$$b_0 = \frac{1}{\pi}\int_0^\pi f(x)dx, \tag{3}$$

which is just half the value that would be given by formula (2) if zero were substituted for m.

To save a separate formula (1) is usually written

$$f(x) = \tfrac{1}{2}b_0 + b_1 \cos x + b_2 \cos 2x + b_3 \cos 3x + \ldots, \tag{4}$$

and then the formula (2) will give b_0 as well as the other coefficients.

Prob. 5. Show that for values of x between 0 and π

(a) $x = \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\right);$

(b) $f(x) = \frac{\pi}{4} - \frac{8}{\pi}\left(\frac{\cos 2x}{2^2} + \frac{\cos 6x}{6^2} + \frac{\cos 10x}{10^2} + \ldots\right),$

if $f(x) = x$ for $0 < x < \frac{\pi}{2}$, and $f(x) = \pi - x$ for $\frac{\pi}{2} < x \not< \pi$;

(c) $f(x) = \frac{1}{2} + \frac{2}{\pi}\left(\frac{\cos x}{1} - \frac{\cos 3x}{3} + \frac{\cos 5x}{5} - \ldots\right),$

if $f(x) = 1$ for $0 < x < \frac{\pi}{2}$, and $f(x) = 0$ for $\frac{\pi}{2} < x < \pi$.

(d) $\sinh x = \frac{2}{\pi}\Big[\frac{1}{2}(\cosh \pi - 1) - \frac{1}{2}(\cosh \pi + 1)\cos x$

$+ \frac{1}{5}(\cosh \pi - 1)\cos 2x - \frac{1}{10}(\cosh \pi + 1)\cos 3x + \ldots\Big];$

(e) $x^2 = \frac{\pi^2}{3} - 4\left(\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \frac{\cos 4x}{4^2} + \ldots\right).$

ART. 8. FOURIER'S SERIES.

Since a sine series is an odd function of x the development of an odd function of x in such a series must hold good from $x = -\pi$ to $x = \pi$, except perhaps for the value $x = 0$, where it is easily seen that the series is necessarily zero, no matter what the value of the function. In like manner we see that if $f(x)$ is an even function of x its development in a cosine series must be valid from $x = -\pi$ to $x = \pi$.

Any function of x can be developed into a Trigonometric series to which it is equal for all values of x between $-\pi$ and π.

Let $f(x)$ be the given function of x. It can be expressed as the sum of an even function of x and an odd function of x by the following device:

$$f(x) = \frac{f(x)+f(-x)}{2} + \frac{f(x)-f(-x)}{2} \tag{1}$$

identically; but $\dfrac{f(x)+f(-x)}{2}$ is not changed by reversing the sign of x and is therefore an even function of x; and when we reverse the sign of x, $\dfrac{f(x)-f(-x)}{2}$ is affected only to the extent of having its sign reversed, and is consequently an odd function of x.

Therefore for all values of x between $-\pi$ and π

$$\frac{f(x)+f(-x)}{2} = \frac{1}{2}b_0 + b_1 \cos x + b_2 \cos 2x + b_3 \cos 3x + \ldots$$

where
$$b_m = \frac{2}{\pi}\int_0^\pi \frac{f(x)+f(-x)}{2} \cos mx \, . \, dx;$$

and
$$\frac{f(x)-f(-x)}{2} = a_1 \sin x + a_2 \sin 2x + a_3 \sin 3x + \ldots$$

where
$$a_m = \frac{2}{\pi}\int_0^\pi \frac{f(x)-f(-x)}{2} \sin mx \, . \, dx.$$

b_m and a_m can be simplified a little.

$$b_m = \frac{2}{\pi}\int_0^{\pi} \frac{f(x) + f(-x)}{2} \cos mx \,.\, dx,$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} f(x) \cos mx \,.\, dx + \int_0^{\pi} f(-x) \cos mx \,.\, dx\right];$$

but if we replace x by $-x$, we get

$$\int_0^{\pi} f(-x) \cos mx \,.\, dx = -\int_0^{-\pi} f(x) \cos mx \,.\, dx = \int_{-\pi}^{0} f(x) \cos mx \,.\, dx,$$

and we have $$b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos mx \,.\, dx.$$

In the same way we can reduce the value of a_m to

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin mx \,.\, dx.$$

Hence

$$f(x) = \frac{1}{2} b_0 + b_1 \cos x + b_2 \cos 2x + b_3 \cos 3x + \dots$$
$$+ a_1 \sin x + a_2 \sin 2x + a_3 \sin 3x + \dots, \quad (2)$$

where $$b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos mx \,.\, dx, \quad (3)$$

and $$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin mx \,.\, dx, \quad (4)$$

and this development holds for all values of x between $-\pi$ and π.

The second member of (2) is known as a Fourier's Series.

The developments of Arts. 5 and 7 are special cases of development in Fourier's Series.

Prob. 6. Show that for all values of x from $-\pi$ to π

$$e^x = \frac{2 \sinh \pi}{\pi}\left[\frac{1}{2} - \frac{1}{2}\cos x + \frac{1}{5}\cos 2x - \frac{1}{10}\cos 3x + \frac{1}{17}\cos 4x + \dots\right]$$

$$+\frac{2\sinh\pi}{\pi}\left[\frac{1}{2}\sin x-\frac{2}{5}\sin 2x+\frac{3}{10}\sin 3x-\frac{4}{17}\sin 4x+\ldots\right].$$

Prob. 7. Show that formula (2), Art. 8, can be written

$$f(x)=\frac{1}{2}c_0\cos\beta_0+c_1\cos(x-\beta_1)+c_2\cos(2x-\beta_2)$$
$$+c_3\cos(3x-\beta_3)+\ldots,$$

where $\qquad c_m=(a_m^2+b_m^2)^{\frac{1}{2}}$ and $\beta_m=\tan^{-1}\frac{a_m}{b_m}$.

Prob. 8. Show that formula (2), Art. 8, can be written

$$f(x)=\frac{1}{2}c_0\sin\beta_0+c_1\sin(x+\beta_1)+c_2\sin(2x+\beta_2)$$
$$+c_3\sin(3x+\beta_3)+\ldots,$$

where $\qquad c_m=(a_m^2+b_m^2)^{\frac{1}{2}}$ and $\beta_m=\tan^{-1}\frac{b_m}{a_m}$.

ART. 9. EXTENSION OF FOURIER'S SERIES.

In developing a function of x into a Trigonometric Series it is often inconvenient to be held within the narrow boundaries $x=-\pi$ and $x=\pi$. Let us see if we cannot widen them.

Let it be required to develop a function of x into a Trigonometric Series which shall be equal to $f(x)$ for all values of x between $x=-c$ and $x=c$.

Introduce a new variable

$$z=\frac{\pi}{c}x,$$

which is equal to $-\pi$ when $x=-c$, and to π when $x=c$.

$f(x)=f\left(\frac{c}{\pi}z\right)$ can be developed in terms of z by Art. 8, (2), (3), and (4). We have

$$f\left(\frac{c}{\pi}z\right)=\frac{1}{2}b_0+b_1\cos z+b_2\cos 2z+b_3\cos 3z+\ldots$$
$$+a_1\sin z+a_2\sin 2z+a_3\sin 3z+\ldots,\qquad(1)$$

where $$b_m=\frac{1}{\pi}\int^{\pi}f\left(\frac{c}{\pi}z\right)\cos mz\,.\,dz,\qquad(2)$$

and $$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f\left(\frac{c}{\pi}z\right)\sin mz\,.\,dz, \tag{3}$$

and (1) holds good from $z = -\pi$ to $z = \pi$.

Replace z by its value in terms of x and (1) becomes

$$f(x) = \frac{1}{2}b_0 + b_1\cos\frac{\pi x}{c} + b_2\cos\frac{2\pi x}{c} + b_3\cos\frac{3\pi x}{c} + \ldots$$
$$+ a_1\sin\frac{\pi x}{c} + a_2\sin\frac{2\pi x}{c} + a_3\sin\frac{3\pi x}{c} + \ldots; \tag{4}$$

and (2) and (3) can be transformed into

$$b_m = \frac{1}{c}\int_{-c}^{c} f(x)\cos\frac{m\pi x}{c}dx, \tag{5}$$

$$a_m = \frac{1}{c}\int_{-c}^{c} f(x)\sin\frac{m\pi x}{c}dx, \tag{6}$$

and (4) holds good from $x = -c$ to $x = c$.

In the formulas just obtained c may have as great a value as we please so that we can obtain a Trigonometric Series for $f(x)$ that will be equal to the given function through as great an interval as we may choose to take.

It can be shown that if this interval c is increased indefinitely the series will approach as its limiting form the double integral $\frac{1}{\pi}\int_{-\infty}^{\infty} f(\lambda)d\lambda\int_0^{\infty}\cos\alpha(\lambda - x)d\alpha$, which is known as a Fourier's Integral. So that

$$f(x) = \frac{1}{\pi}\int_{-\infty}^{+\infty} f(\lambda)d\lambda\int_0^{\infty}\cos\alpha(\lambda - x)d\alpha \tag{7}$$

for all values of x.

For the treatment of Fourier's Integral and for examples of its use in Mathematical Physics the student is referred to Riemann's Partielle Differentialgleichungen, to Schlömilch's Höhere Analysis, and to Byerly's Fourier's Series and Spherical Harmonics.

Prob. 9. Show that formula (4), Art. 9, can be written

$$f(x) = \frac{1}{2} c_0 \cos \beta_0 + c_1 \cos \left(\frac{\pi x}{c} - \beta_1\right) + c_2 \cos \left(\frac{2\pi x}{c} - \beta_2\right) + c_3 \cos \left(\frac{3\pi x}{c} - \beta_3\right) + \ldots,$$

where $c_m = (a_m^2 + b_m^2)^{\frac{1}{2}}$ and $\beta_m = \tan^{-1} \frac{a_m}{b_m}$.

Prob. 10. Show that formula (4), Art. 9, can be written

$$f(x) = \frac{1}{2} c_0 \sin \beta_0 + c_1 \sin \left(\frac{\pi x}{c} + \beta_1\right) + c_2 \sin \left(\frac{2\pi x}{c} + \beta_2\right) + c_3 \sin \left(\frac{3\pi x}{c} + \beta_3\right) + \ldots,$$

where $c_m = (a_m^2 + b_m^2)^{\frac{1}{2}}$ and $\beta_m = \tan^{-1} \frac{b_m}{a_m}$.

ART. 10. DIRICHLET'S CONDITIONS.

In determining the coefficients of the Fourier's Series representing $f(x)$ we have virtually assumed, first, that a series of the required form and equal to $f(x)$ exists; and second, that it is *uniformly convergent;* and consequently we must regard the results obtained as only provisionally established.

It is, however, possible to prove rigorously that if $f(x)$ is finite and single-valued from $x = -\pi$ to $x = \pi$ and has not an infinite number of (finite) discontinuities, or of maxima or minima between $x = -\pi$ and $x = \pi$, the Fourier's Series of (2), Art. 8, and that Fourier's Series only, is equal to $f(x)$ for all values of x between $-\pi$ and π, excepting the values of x corresponding to the discontinuities of $f(x)$, and the values π and $-\pi$; and that if c is a value of x corresponding to a discontinuity of $f(x)$, the value of the series when $x = c$ is $\frac{1}{2} \lim_{\epsilon=0} [f(c+\epsilon) + f(c-\epsilon)]$; and that when $x = \pi$ or $x = -\pi$ the value of the series is $\frac{1}{2}[f(\pi) + f(-\pi)]$.

This proof was first given by Dirichlet in 1829, and may be found in readable form in Riemann's Partielle Differentialgleichungen and in Picard's Traité d'Analyse, Vol. I.

A good deal of light is thrown on the peculiarities of trigonometric series by the attempt to construct approximately the curves corresponding to them.

If we construct $y = a_1 \sin x$ and $y = a_2 \sin 2x$ and add the ordinates of the points having the same abscissas, we shall obtain points on the curve

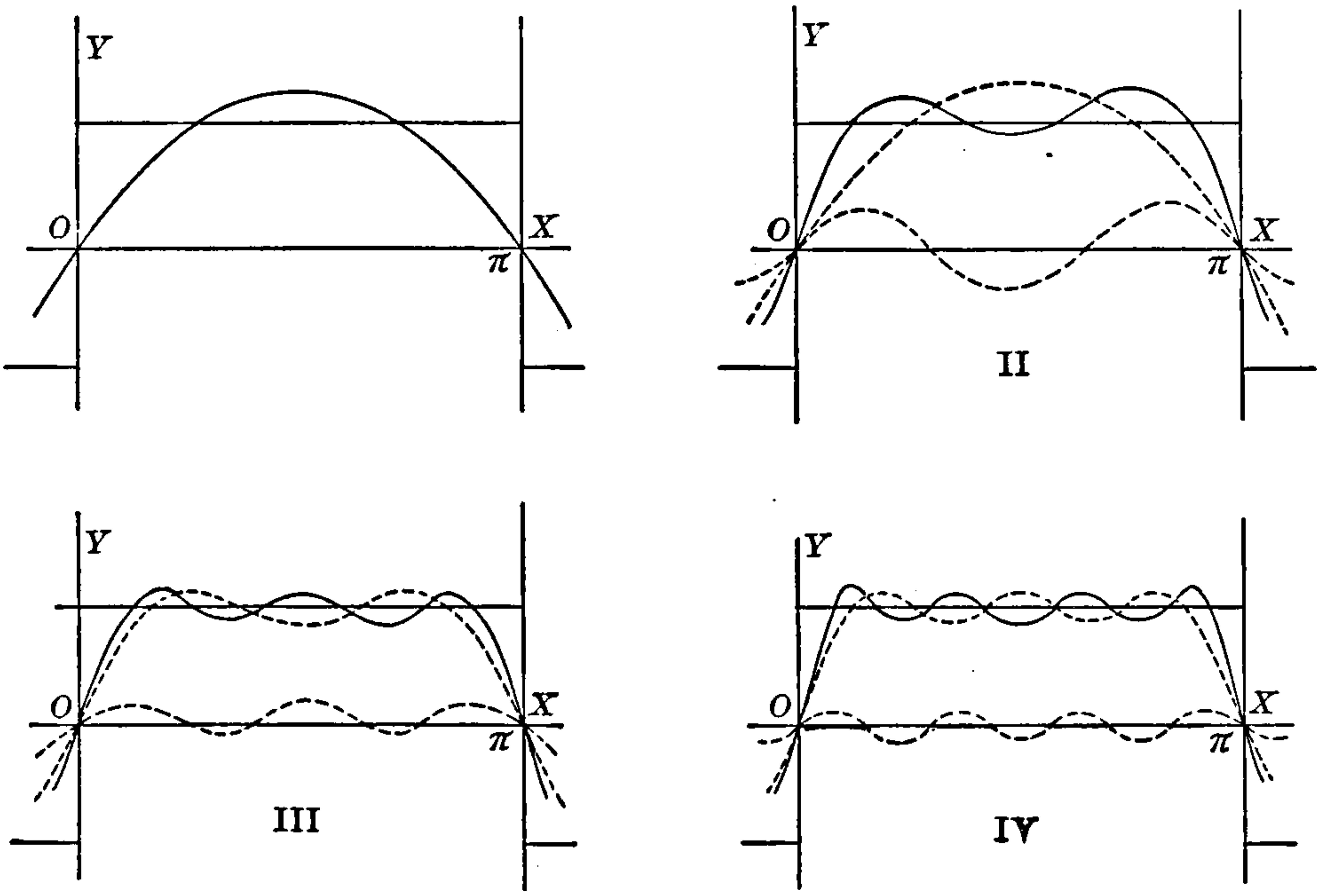
$$y = a_1 \sin x + a_2 \sin 2x.$$

If now we construct $y = a_3 \sin 3x$ and add the ordinates to those of $y = a_1 \sin x + a_2 \sin 2x$ we shall get the curve

$$y = a_1 \sin x + a_2 \sin 2x + a_3 \sin 3x.$$

By continuing this process we get successive approximations to

$$y = a_1 \sin x + a_2 \sin 2x + a_3 \sin 3x + a_4 \sin 4x + \cdots$$



Let us apply this method to the series

$y = \sin x + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x + \cdots$ (1) (See (7), Art. 6.)

$y = 0$ when $x = 0$, $\dfrac{\pi}{4}$ from $x = 0$ to $x = \pi$, and 0 when $x = \pi$.

It must be borne in mind that our curve is periodic, having the period 2π, and is symmetrical with respect to the origin.

The preceding figures represent the first four approxima-

tion to this curve. In each figure the curve $y =$ the series, and the approximations in question are drawn in continuous lines, and the preceding approximation and the curve corresponding to the term to be added are drawn in dotted lines.

Prob. 11. Construct successive approximations to the series given in the examples at the end of Art. 6.

Prob. 12. Construct successive approximations to the Maclaurin's Series for sinh x, namely $x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$

ART. 11. APPLICATIONS OF TRIGONOMETRIC SERIES.

(*a*) Three edges of a rectangular plate of tinfoil are kept at potential zero, and the fourth at potential 1. At what potential is any point in the plate?

Here we have to solve Laplace's Equation (3), Art. 1, which, since the problem is two-dimensional, reduces to

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0, \tag{1}$$

subject to the conditions $V = 0$ when $x = 0$, (2)

$$V = 0 \quad \text{“} \quad x = a, \tag{3}$$

$$V = 0 \quad \text{“} \quad y = 0, \tag{4}$$

$$V = 1 \quad \text{“} \quad y = b. \tag{5}$$

Working as in Art. 3, we readily get $\sinh \beta y \sin \beta x$, $\sinh \beta y \cos \beta x$, $\cosh \beta y \sin \beta x$, and $\cosh \beta y \cos \beta x$ as particular values of V satisfying (1).

$V = \sinh \frac{m\pi y}{a} \sin \frac{m\pi x}{a}$ satisfies (1), (2), (3), and (4).

$$V = \frac{4}{\pi}\left[\frac{\sinh \frac{\pi y}{a}}{\sinh \frac{\pi b}{a}} \sin \frac{\pi x}{a} + \frac{1}{3}\frac{\sinh \frac{3\pi y}{a}}{\sinh \frac{3\pi b}{a}} \sin \frac{3\pi x}{a} + \cdots\right] \tag{6}$$

is the required solution, for it reduces to 1 when $y = b$. See (7), Art. 6.

(*b*) A harp-string is initially distorted into a given plane curve and then released; find its motion.

The differential equation for the small transverse vibrations of a stretched elastic string is

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}, \tag{1}$$

as stated in Art. 1. Our conditions if we take one end of the string as origin are

$$y = 0 \text{ when } x = 0, \tag{2}$$
$$y = 0 \quad \text{"} \quad x = l, \tag{3}$$
$$\frac{\partial y}{\partial t} = 0 \quad \text{"} \quad t = 0, \tag{4}$$
$$y = fx \quad \text{"} \quad t = 0. \tag{5}$$

Using the method of Art. 3, we easily get as particular solutions of (1)

$$y = \sin \beta x \sin a\beta t, \qquad y = \sin \beta x \cos a\beta t,$$
$$y = \cos \beta x \sin a\beta t, \quad \text{and} \quad y = \cos \beta x \cos a\beta t.$$

$$y = \sin \frac{m\pi x}{l} \cos \frac{m\pi at}{l} \text{ satisfies (1), (2), (3), and (4).}$$

$$y = \sum_{m=1}^{m=\infty} a_m \sin \frac{m\pi x}{l} \cos \frac{m\pi at}{l}, \tag{6}$$

where
$$a_m = \frac{2}{l} \int_0^l f(x) \sin \frac{m\pi x}{l} . dx \tag{7}$$

is our required solution; for it reduces to $f(x)$ when $t = 0$. See Art. 9.

Prob. 13. Three edges of a square sheet of tinfoil are kept at potential zero, and the fourth at potential unity; at what potential is the centre of the sheet? Ans. 0.25.

Prob. 14. Two opposite edges of a square sheet of tinfoil are kept at potential zero, and the other two at potential unity; at what potential is the centre of the sheet? Ans. 0.5.

Prob. 15. Two adjacent edges of a square sheet of tinfoil are

kept at potential zero, and the other two at potential unity. At what potential is the centre of the sheet? Ans. 0.5.

Prob. 16. Show that if a point whose distance from the end of a harp-string is $\frac{1}{n}$th the length of the string is drawn aside by the player's finger to a distance b from its position of equilibrium and then released, the form of the vibrating string at any instant is given by the equation

$$y = \frac{2bn^2}{(n-1)\pi^2} \sum_{m=1}^{m=\infty} \left(\frac{1}{m^2} \sin \frac{m\pi}{n} \sin \frac{m\pi x}{l} \cos \frac{m\pi at}{l} \right).$$

Show from this that all the harmonics of the fundamental note of the string which correspond to forms of vibration having nodes at the point drawn aside by the finger will be wanting in the complex note actually sounded.

Prob. 17.* An iron slab 10 centimeters thick is placed between and in contact with two other iron slabs each 10 centimeters thick. The temperature of the middle slab is at first 100 degrees Centigrade throughout, and of the outside slabs zero throughout. The outer faces of the outside slabs are kept at the temperature zero. Required the temperature of a point in the middle of the middle slab fifteen minutes after the slabs have been placed in contact. Given $a^2 = 0.185$ in C.G.S. units. Ans. $10^\circ.3$.

Prob. 18.* Two iron slabs each 20 centimeters thick, one of which is at the temperature zero and the other at 100 degrees Centigrade throughout, are placed together face to face, and their outer faces are kept at the temperature zero. Find the temperature of a point in their common face and of points 10 centimeters from the common face fifteen minutes after the slabs have been put together.

Ans. $22^\circ.8$; $15^\circ.1$; $17^\circ.2$.

ART. 12.† PROPERTIES OF ZONAL HARMONICS.

In Art. 4, $z = P_m(x)$ was obtained as a particular solution of Legendre's Equation [(7), Art. 4] by the device of assuming that z could be expressed as a sum or a series of terms of the form $a_n x^n$ and then determining the coefficients. We

* See Art. 3.

† The student should review Art. 4 before beginning this article.

can, however, obtain a particular solution of Legendre's equation by an entirely different method.

The potential function for any point (x, y, z) due to a unit of mass concentrated at a given point (x_1, y_1, z_1) is

$$V = \frac{1}{\sqrt{(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2}}, \tag{1}$$

and this must be a particular solution of Laplace's Equation [(3), Art. 1], as is easily verified by direct substitution.

If we transform (1) to spherical coordinates we get

$$V = \frac{1}{\sqrt{r^2 - 2rr_1[\cos\theta\cos\theta_1 + \sin\theta\sin\theta_1\cos(\phi - \phi_1)] + r_1^2}} \tag{2}$$

as a solution of Laplace's Equation in Spherical Coordinates [(5), Art. 1].

If the given point (x_1, y_1, z_1) is taken on the axis of X, as it must be in order that (2) may be independent of ϕ, $\theta_1 = 0$, and

$$V = \frac{1}{\sqrt{r^2 - 2rr_1\cos\theta + r_1^2}} \tag{3}$$

is a solution of equation (1), Art. 4.

Equation (3) can be written

$$V = \frac{1}{r_1}\left(1 - 2\frac{r}{r_1}\cos\theta + \frac{r^2}{r_1^2}\right)^{-\frac{1}{2}}; \tag{4}$$

and if r is less than r_1 $\left(1 - 2\frac{r}{r_1}\cos\theta + \frac{r^2}{r_1^2}\right)^{-\frac{1}{2}}$ can be developed into a convergent power series. Let $\Sigma p_m \frac{r^m}{r_1^m}$ be this series, p_m being of course a function of θ. Then $V = \frac{1}{r_1}\Sigma p_m \frac{r^m}{r_1^m}$ is a solution of (1), Art. 4.

Substituting this value of V in the equation, and remembering that the result must be identically true, we get after a slight reduction

$$m(m + 1)p_m + \frac{1}{\sin\theta}\frac{d}{d\theta}\left[\sin\theta\frac{dp_m}{d\theta}\right] = 0;$$

but, as we have seen, the substitution of $x = \cos\theta$ reduces this to Legendre's equation [(7), Art. 4]. Hence we infer that the coefficient of the mth power of z in the development of $(1-2xz+z^2)^{-\frac{1}{2}}$ is a function of x that will satisfy Legendre's equation.

$$(1-2xz+z^2)^{-\frac{1}{2}} = [1-z(2x-z)]^{-\frac{1}{2}},$$

and can be developed by the Binomial Theorem; the coefficient of z^m is easily picked out, and proves to be precisely the function of x which in Art. 4 we have represented by $P_m(x)$, and have called a Surface Zonal Harmonic.

We have, then,

$$(1-2xz+z^2)^{-\frac{1}{2}} = P_0(x) + P_1(x).z + P_2(x).z^2 + P_3(x).z^3 + \ldots \quad (5)$$

if the absolute value of z is less than 1.

If $x = 1$, (5) reduces to

$$(1-2z+z^2)^{-\frac{1}{2}} = P_0(1) + P_1(1).z + P_2(1).z^2 + P_3(1).z^3 + \ldots;$$

but $(1-2z+z^2)^{-\frac{1}{2}} = (1-z)^{-1} = 1 + z + z^2 + z^3 + \ldots;$

hence $$P_m(1) = 1. \quad (6)$$

Any Surface Zonal Harmonic may be obtained from the two of next lower orders by the aid of the formula

$$(n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x) = 0, \quad (7)$$

which is easily obtained, and is convenient when the numerical value of x is given.

Differentiate (5) with respect to z, and we get

$$\frac{-(z-x)}{(1-2xz+z^2)^{\frac{3}{2}}} = P_1(x) + 2P_2(x).z + 3P_3(x).z^2 + \ldots,$$

whence

$$\frac{-(z-x)}{(1-2xz+z^2)^{\frac{1}{2}}} = (1-2xz+z^2)(P_1(x) + 2P_2(x).z + 3P_3(x).z^2 + \ldots),$$

or by (5)

$$(1-2xz+z^2)(P_1(x) + 2P_2(x).z + 3P_3(x).z^2 \ldots)$$
$$+ (z-x)(P_0(x) + P_1(x).z + P_2(x).z^2 + \ldots) = 0. \quad (8)$$

Now (8) is identically true, hence the coefficient of each power of z must vanish. Picking out the coefficient of z^n and writing it equal to zero, we have formula (7) above.

By the aid of (7) a table of Zonal Harmonics is easily computed since we have $P_0(x) = 1$, and $P_1(x) = x$. Such a table for $x = \cos\theta$ is given at the end of this chapter.

ART. 13. PROBLEMS IN ZONAL HARMONICS.

In any problem on Potential if V is independent of ϕ so that we can use the form of Laplace's Equation employed in Art. 4, and if the value of V on the axis of X is known, and can be expressed as $\Sigma a_m r^m$ or as $\sum \frac{b_m}{r^{m+1}}$, we can write out our required solution as

$$V = \Sigma a_m r^m P_m(\cos\theta) \quad \text{or} \quad V = \sum \frac{b_m P_m(\cos\theta)}{r^{m+1}};$$

for since $P_m(1) = 1$ each of these forms reduces to the proper value on the axis; and as we have seen in Art. 4 each of them satisfies the reduced form of Laplace's Equation.

As an example, let us suppose a statical charge of M units of electricity placed on a conductor in the form of a thin circular disk, and let it be required to find the value of the Potential Function at any point in the "field of force" due to the charge.

The surface density at a point of the plate at a distance r from its centre is

$$\sigma = \frac{M}{4a\pi\sqrt{a^2 - r^2}}$$

and all points of the conductor are at potential $\frac{\pi M}{2a}$. See Pierce's Newtonian Potential Function (§ 61).

The value of the potential function at a point in the axis of the plate at the distance x from the plate can be obtained without difficulty by a simple integration, and proves to be

$$V = \frac{M}{2a}\cos^{-1}\frac{x^2 - a^2}{x^2 + a^2}. \tag{1}$$

The second member of (1) is easily developed into a power series.

$$\frac{M}{2a}\cos^{-1}\frac{x^2-a^2}{x^2+a^2}$$

$$=\frac{M}{a}\left[\frac{\pi}{2}-\frac{x}{a}+\frac{x^3}{3a^3}-\frac{x^5}{5a^5}+\frac{x^7}{7a^7}-\dots\right] \text{ if } x<a \quad (2)$$

$$=\frac{M}{a}\left[\frac{a}{x}-\frac{a^3}{3x^3}+\frac{a^5}{5x^5}-\frac{a^7}{7x^7}+\dots\right] \text{ if } x>a. \quad (3)$$

Hence

$$V=\frac{M}{a}\left[\frac{\pi}{2}-\frac{r}{a}P_1(\cos\theta)+\frac{1}{3}\frac{r^3}{a^3}P_3(\cos\theta)-\frac{1}{5}\frac{r^5}{a^5}P_5(\cos\theta)+\dots\right] \quad (4)$$

is our required solution if $r<a$ and $\theta<\frac{\pi}{2}$, as is

$$V=\frac{M}{a}\left[\frac{a}{r}-\frac{1}{3}\frac{a^3}{r^3}P_2(\cos\theta)+\frac{1}{5}\frac{a^5}{r^5}P_4(\cos\theta)-\frac{1}{7}\frac{a^7}{r^7}P_6(\cos\theta)+\dots\right] \text{ if } r>a. \quad (5)$$

The series in (4) and (5) are convergent, since they may be obtained from the convergent series (2) and (3) by multiplying the terms by a set of quantities no one of which exceeds one in absolute value. For it will be shown in the next article that $P_m(\cos\theta)$ always lies between 1 and -1.

Prob. 19. Find the value of the Potential Function due to the attraction of a material circular ring of small cross-section.

The value on the axis of the ring can be obtained by a simple integration, and is $\dfrac{M}{\sqrt{c^2+r^2}}$ if M is the mass and c the radius of the ring. At any point in space, if $r<c$

$$V=\frac{M}{c}\left[P_0(\cos\theta)-\frac{1}{2}\frac{r^2}{c^2}P_2(\cos\theta)+\frac{1\cdot 3}{2\cdot 4}\frac{r^4}{c^4}P_4(\cos\theta)-\dots\right],$$

and if $r>c$

$$V=\frac{M}{c}\left[\frac{c}{r}P_0(\cos\theta)-\frac{1}{2}\frac{c^3}{r^3}P_2(\cos\theta)+\frac{1\cdot 3}{2\cdot 4}\frac{c^5}{r^5}P_4(\cos\theta)-\dots\right].$$

ART. 14. ADDITIONAL FORMS.

(*a*) We have seen in Art. 12 that $P_m(x)$ is the coefficient of z^m in the development of $(1-2xz+z^2)^{-\frac{1}{2}}$ in a power series.

$$\begin{aligned}(1-2xz+z^2)^{-\frac{1}{2}}&=[1-z(e^{\theta i}+e^{-\theta i})+z^2]^{-\frac{1}{2}}\\&=(1-ze^{\theta i})^{-\frac{1}{2}}(1-ze^{-\theta i})^{-\frac{1}{2}}.\end{aligned}$$

If we develop $(1-ze^{\theta i})^{-\frac{1}{2}}$ and $(1-ze^{-\theta i})^{-\frac{1}{2}}$ by the Binomial Theorem their product will give a development for $(1-2xz+z^2)^{-\frac{1}{2}}$. The coefficient of z^m is easily picked out and reduced, and we get

$$P_m(\cos\theta)=$$

$$\frac{1\cdot3\cdot5\dots(2m-1)}{2\cdot4\cdot6\dots2m}\left[2\cos m\theta+2\frac{1\cdot m}{1\cdot(2m-1)}\cos(m-2)\theta\right.$$

$$\left.+2\frac{1\cdot3\cdot m(m-1)}{1\cdot2\cdot(2m-1)(2m-3)}\cos(m-4)\theta+\dots\right] \qquad (1)$$

If m is odd the parenthesis in (1) ends with the term containing $\cos\theta$; if m is even, with the term containing cos 0, but in the latter case the term in question will not be multiplied by the factor 2, which is common to all the other terms.

Since all the coefficients in the second member of (1) are positive, $P_m(\cos\theta)$ has its maximum value when $\theta=0$, and its value then has already been shown in Art. 12 to be unity. Obviously, then, its minimum value cannot be less than -1.

(*b*) If we integrate the value of $P_m(x)$ given in (11), Art. 4, m times in succession with respect to x, the result will be found to differ from $\frac{1\cdot3\cdot5\dots(2m-1)}{(2m)!}(x^2-1)^m$ by terms involving lower powers of x than the mth.

Hence $$P_m(x)=\frac{1}{2^m m!}\frac{d^m}{dx^m}(x^2-1)^m. \qquad (2)$$

(*c*) Other forms for $P_m(x)$, which we give without demonstration, are

$$P_m(x) = \frac{(-1)^m}{m!}\frac{\partial^m}{\partial x^m}\frac{1}{\sqrt{x^2+y^2+z^2}}. \tag{3}$$

$$P_m(x) = \frac{1}{\pi}\int_0^\pi [x+\sqrt{x^2-1}\,.\cos\phi]^m d\phi. \tag{4}$$

$$P_m(x) = \frac{1}{\pi}\int_0^\pi \frac{d\phi}{[x-\sqrt{x^2-1}\,.\cos\phi]^{m+1}}. \tag{5}$$

(4) and (5) can be verified without difficulty by expanding and integrating.

ART. 15. DEVELOPMENT IN TERMS OF ZONAL HARMONICS.

Whenever, as in Art. 4, we have the value of the Potential Function given on the surface of a sphere, and this value depends only on the distance from the extremity of a diameter, it becomes necessary to develop a function of θ into a series of the form

$$A_0P_0(\cos\theta) + A_1P_1(\cos\theta) + A_2P_2(\cos\theta) + \ldots;$$

or, what amounts to the same thing, to develop a function of x into a series of the form

$$A_0P_0(x) + A_1P_1(x) + A_2P_2(x) + \ldots.$$

The problem is entirely analogous to that of development in sine-series treated at length in Art. 6, and may be solved by the same method.

Assume $f(x) = A_0P_0(x) + A_1P_1(x) + A_2P_2(x) + \ldots$ (1)

for $-1 < x < 1$. Multiply (1) by $P_m(x)dx$ and integrate from -1 to 1. We get

$$\int_{-1}^{1} f(x)P_m(x)dx = \sum_{n=0}^{n=\infty}\left[A_n\int_{-1}^{1}P_m(x)P_n(x)dx\right]. \tag{2}$$

We shall show in the next article that

$$\int_{-1}^{1} P_m(x)P_n(x)dx = 0, \quad \text{unless } m = n,$$

and that $\displaystyle\int_{-1}^{1}[P_m(x)]^2 dx = \frac{2}{2m+1}.$

Hence
$$A_m = \frac{2m+1}{2}\int_{-1}^{1} f(x)P_m(x)dx. \qquad (3)$$

It is important to notice here, as in Art. 6, that the method we have used in obtaining A_m amounts essentially to determining A_m, so that the equation

$$f(x) = A_0P_0(x) + A_1P_1(x) + A_2P_2(x) + \ldots + A_nP_n(x)$$

shall hold good for $n+1$ equidistant values of x between -1 and 1, and taking its limiting value as n is indefinitely increased.

ART. 16. FORMULAS FOR DEVELOPMENT.

We have seen in Art. 4 that $z = P_m(x)$ is a solution of Legendre's Equation $\displaystyle\frac{d}{dx}\left[(1-x^2)\frac{dz}{dx}\right] + m(m+1)z = 0.$ (1)

Hence
$$\frac{d}{dx}\left[(1-x^2)\frac{dP_m(x)}{dx}\right] + m(m+1)P_m(x) = 0, \qquad (2)$$

and
$$\frac{d}{dx}\left[(1-x^2)\frac{dP_n(x)}{dx}\right] + n(n+1)P_n(x) = 0. \qquad (3)$$

Multiply (2) by $P_n(x)$ and (3) by $P_m(x)$, subtract, transpose, and integrate. We have

$$[m(m+1) - n(n+1)]\int_{-1}^{1} P_m(x)P_n(x)dx$$

$$\int_{-1}^{1} P_m(x)\frac{d}{dx}\left[(1-x^2)\frac{dP_n(x)}{dx}\right]dx$$

$$-\int_{-1}^{1} P_n(x)\frac{d}{dx}\left[(1-x^2)\frac{dP_m(x)}{dx}\right]dx \qquad (4)$$

$$= \left[P_m(x)(1-x^2)\frac{dP_n(x)}{dx} - P_n(x)(1-x^2)\frac{dP_m(x)}{dx}\right]_{-1}^{1}$$

$$-\int_{-1}^{1}(1-x^2)\frac{dP_n(x)}{dx}\,\frac{dP_m(x)}{dx}\,.\,dx$$

$$+\int_{-1}^{1}(1-x^2)\frac{dP_m(x)}{dx}\,\frac{dP_n(x)}{dx}\,.\,dx \qquad (5)$$

by integration by parts,

$= 0.$

Hence
$$\int_{-1}^{1} P_m(x)P_n(x)dx = 0, \qquad (6)$$
unless $m = n$.

If in (4) we integrate from x to 1 instead of from -1 to 1, we get an important formula.

$$\int_{x}^{1} P_m(x)P_n(x)dx = \frac{(1-x^2)\left[P_n(x)\dfrac{dP_m(x)}{dx} - P_m(x)\dfrac{dP_n(x)}{dx}\right]}{m(m+1)-n(n+1)}, \qquad (7)$$

and as a special case, since $P_0(x) = 1$.

$$\int_{x}^{1} P_m(x)dx = \frac{(1-x^2)\dfrac{dP_m(x)}{dx}}{m(m+1)}, \qquad (8)$$

unless $m = 0$.

To get $\int_{-1}^{1}[P_m(x)]^2dx$ is not particularly difficult. By (2), Art. 14,

$$\int_{-1}^{1}[P_m(x)]^2dx = \frac{1}{2^{2m}(m!)^2}\int_{-1}^{1}\frac{d^m(x^2-1)^m}{dx^m}\,.\,\frac{d^m(x^2-1)^m}{dx^m}\,.\,dx \qquad (9)$$

By successive integrations by parts, noting that $\dfrac{d^{m-\kappa}}{dx^{m-\kappa}}(x^2-1)^m$ contains $(x^2-1)^\kappa$ as a factor if $\kappa < m$, and

that $\dfrac{d^{2m}(x^2-1)^m}{dx^{2m}} = (2m)!$ we get

$$\int_{-1}^{1}[P_m(x)]^2dx = \frac{(-1)^m(2m)!}{2^{2m}(m!)^2}\int_{-1}^{1}(x^2-1)^m dx. \tag{10}$$

$$\int_{-1}^{1}(x^2-1)^m dx = \int_{-1}^{1}(x-1)^m(x+1)^m dx$$

$$= -\frac{m}{m+1}\int_{-1}^{1}(x-1)^{m-1}(x+1)^{m+1}dx$$

$$= (-1)^m\frac{m!\,m!}{(2m)!}\int_{-1}^{1}(x+1)^{2m}dx = (-1)^m\frac{2^{2m+1}(m!)^2}{(2m+1)!}.$$

Hence
$$\int_{-1}^{1}[P_m(x)]^2dx = \frac{2}{2m+1}. \tag{11}$$

Prob. 20. Show that $\int_0^1 P_m(x)dx = 0$ if m is even and is not zero

$$= (-1)^{\frac{m-1}{2}}\frac{1}{m(m+1)}\cdot\frac{3\cdot5\cdot7\ldots m}{2\cdot4\cdot6\ldots(m-1)} \text{ if } m \text{ is odd.}$$

Prob. 21. Show that $\int_0^1[P_m(x)]^2dx = \dfrac{1}{2m+1}$. Note that $[P_m(x)]^2$ is an even function of x.

Prob. 22. Show that if $f(x)=0$ from $x=-1$ to $x=0$, and $f(x)=1$ from $x=0$ to $x=1$,

$$f(x) = \frac{1}{2}+\frac{3}{4}P_1(x)-\frac{7}{8}\cdot\frac{1}{2}P_3(x)+\frac{11}{12}\cdot\frac{1\cdot3}{2\cdot4}P_5(x)-\ldots$$

Prob. 23. Show that $F(\theta) = \sum\limits_{m=0}^{m=\infty}B_mP_m(\cos\theta)$ where

$$B_m = \frac{2m+1}{2}\int_0^{\pi}F(\theta)P_m(\cos\theta)\sin\theta\,d\theta.$$

Prob. 24. Show that

$$\csc\theta = \frac{\pi}{2}\left[1 + 5\left(\frac{1}{2}\right)^2 P_2(\cos\theta) + 9\left(\frac{1\cdot 3}{2\cdot 4}\right)^2 P_4(\cos\theta) + \dots\right].$$

See (1), Art. 14.

Prob. 25. Show that

$$x^n = \frac{n!}{1\cdot 3\cdot 5\dots(2n+1)}\Big[(2n+1)P_n(x) + (2n-3)\frac{2n+1}{2}P_{n-2}(x) + (2n-7)\frac{(2n+1)(2n-1)}{2\cdot 4}P_{n-4}(x) + \dots .$$

Note that $\int_{-1}^{1} x^n P_m(x)dx = \frac{1}{2^m m!}\int_{-1}^{1} x^n \frac{d^m(x^2-1)^m}{dx^m}dx$, and use the method of integration by parts freely.

Prob. 26. Show that if V is the value of the Potential Function at any point in a field of force, not imbedded in attracting or repelling matter; and if $V = f(\theta)$ when $r = a$,

$$V = \Sigma A_m \frac{r^m}{a^m} P_m(\cos\theta) \text{ if } r < a$$

and

$$V = \Sigma A_m \frac{a^{m+1}}{r^{m+1}} P_m(\cos\theta) \text{ if } r > a,$$

where

$$A_m = \frac{2m+1}{2}\int_0^\pi f(\theta)P_m(\cos\theta)\sin\theta d\theta.$$

Prob. 27. Show that if

$$V = c \text{ when } r = a;\quad V = c \text{ if } r < a, \quad\text{and } V = \frac{ca}{r} \text{ if } r > a.$$

ART. 17. FORMULAS IN ZONAL HARMONICS.

The following formulas which we give without demonstration may be found useful for reference:

$$\frac{dP_n(x)}{dx} = (2n-1)P_{n-1}(x) + (2n-5)P_{n-3}(x) + (2n-9)P_{n-5}(x) + \dots \quad (1)$$

$$\frac{dP_{n+1}(x)}{dx} - \frac{dP_{n-1}(x)}{dx} = (2n+1)P_n(x)$$

$$\int_x^1 P_n(x)dx = \frac{1}{2n+1}[P_{n-1}(x) - P_{n+1}(x)]. \quad (3)$$

ART. 18. SPHERICAL HARMONICS.

In problems in Potential where the value of V is given on the surface of a sphere, but is not independent of the angle ϕ, we have to solve Laplace's Equation in the form (5), Art. 1, and by a treatment analogous to that given in Art. 4 it can be proved that

$$V = r^m \cos n\phi \sin^n \theta \frac{d^n P_m(\mu)}{d\mu_n} \quad \text{and} \quad V = r^m \sin n\phi \sin^n \theta \frac{d^n P_m(\mu)}{d\mu^n},$$

where $\mu = \cos\theta$, are particular solutions of (5), Art. 1.

The factors multiplied by r^m in these values are known as Tesseral Harmonics. They are functions of ϕ and θ, and they play nearly the same part in unsymmetrical problems that the Zonal Harmonics play in those independent of ϕ.

$$Y_m(\mu, \phi) = A_0 P_m(\mu) + \sum_{m=1}^{n=m} (A_n \cos n\phi + B_n \sin n\phi) \sin^n \theta \frac{d^n P_m(\mu)}{d\mu^n}$$

is known as a Surface Spherical Harmonic of the mth degree,

and $\qquad V = r^m Y_m(\mu, \phi)$ and $V = \frac{1}{r^{m+1}} Y_m(\mu, \phi)$

satisfy Laplace's Equation, (5), Art. 1.

The Tesseral and the Zonal Harmonics are special cases of the Spherical Harmonic, as is also a form $P_m(\cos\gamma)$ known as a Laplace's Coefficient or a Laplacian; γ standing for the angle between r and the radius vector r_1 of some fixed point.

For the properties and uses of Spherical Harmonics we refer the student to more extended treatises, namely, to Ferrer's Spherical Harmonics, to Heine's Kugelfunctionen, or to Byerly's Fourier's Series and Spherical Harmonics.

ART. 19.* BESSEL'S FUNCTIONS. PROPERTIES.

We have seen in Art. 5 that $z = J_0(x)$ where

$$J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} - \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \dots \qquad (1)$$

* **The student should review Art. 5 before reading this article.**

is a solution of the equation

$$\frac{d^2z}{dx^2}+\frac{1}{x}\frac{dz}{dx}+z=0; \tag{2}$$

and we have called $J_0(x)$ a Bessel's Function or Cylindrical Harmonic of the zero order.

$$J_1(x)=-\frac{dJ_0(x)}{dx}=\frac{x}{2}\left[1-\frac{x^2}{2.4}+\frac{x^4}{2.4^2.6}-\frac{x^6}{2.4^2.6^2.8}+\ldots\right] \tag{3}$$

is called a Bessel's Function of the first order, and

$$z'=J_1(x)$$

is a solution of the equation

$$\frac{d^2z'}{dx^2}+\frac{1}{x}\frac{dz'}{dx}+\left(1-\frac{1}{x^2}\right)z'=0, \tag{4}$$

which is the result of differentiating (2) with respect to x.

A table giving values of $J_0(x)$ and $J_1(x)$ will be found at the end of this chapter.

If we write $J_0(x)$ for z in equation (2), then multiply through by xdx and integrate from zero to x, simplifying the resulting equation by integration by parts, we get

$$\frac{xdJ_0(x)}{dx}+\int_0^x xJ_0(x)dx=0,$$

or, since $J_1(x)=-\dfrac{dJ_0(x)}{dx}$,

$$\int_0^x xJ_0(x)dx=xJ_1(x). \tag{5}$$

If we write $J_0(x)$ for z in equation (2), then multiply through by $x^2\dfrac{dJ_0(x)}{dx}$, and integrate from zero to x, simplifying by integration by parts, we get

$$\frac{x^2}{2}\left[\left(\frac{dJ_0(x)}{dx}\right)^2+(J_0(x))^2\right]-\int_0^x x(J_0(x))^2dx=0,$$

or

$$\int_0^x x(J_0(x))^2dx=\frac{x^2}{2}\left[(J_0(x))^2+(J_1(x))^2\right]. \tag{6}$$

If we replace x by μx in (2) it becomes

$$\frac{d^2z}{dx^2} + \frac{1}{x}\frac{dz}{dx} + \mu^2 z = 0 \tag{7}$$

(See (8), Art. 5). Hence $z = J_0(\mu x)$ is a solution of (7).

If we substitute in turn in (7) $J_0(\mu_\kappa x)$ and $J_0(\mu_\iota x)$ for z, multiply the first equation by $xJ_0(\mu_\iota x)$, the second by $xJ_0(\mu_\kappa x)$, subtract the second from the first, simplify by integration by parts, and reduce, we get

$$\int_0^a xJ_0(\mu_\kappa x)J_0(\mu_\iota x)dx$$

$$= \frac{1}{\mu_\kappa^2 - \mu_\iota^2}[\mu_\kappa a J_0(\mu_\iota a)J_1(\mu_\kappa a) - \mu_\iota a J_0(\mu_\kappa a)J_1(\mu_\iota a)]. \tag{8}$$

Hence if μ_κ and μ_ι are different roots of $J_0(\mu a) = 0$, or of $J_1(\mu a) = 0$, or of $\mu a J_1(\mu a) - \lambda J_0(\mu a) = 0$,

$$\int_0^a xJ_0(\mu_\kappa x)J_0(\mu_\iota x)dx = 0. \tag{9}$$

We give without demonstration the following formulas, which are sometimes useful:

$$J_0(x) = \frac{1}{\pi}\int_0^\pi \cos(x\cos\phi)d\phi. \tag{10}$$

$$J_1(x) = \frac{x}{\pi}\int_0^\pi \sin^2\phi\cos(x\cos\phi)d\phi. \tag{11}$$

They can be confirmed by developing $\cos(x\cos\phi)$, integrating, and comparing with (1) and (3).

ART. 20. APPLICATIONS OF BESSEL'S FUNCTIONS.

(*a*) The problem of Art. 5 is a special case of the following: The convex surface and one base of a cylinder of radius a and length b are kept at the constant temperature zero, the temperature at each point of the other base is a given function of the distance of the point from the center of the base; re-

quired the temperature of any point of the cylinder after the permanent temperatures have been established.

Here we have to solve Laplace's Equation in the form

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} = 0 \tag{1}$$

(see Art. 5), subject to the conditions

$$u = 0 \text{ when } z = 0,$$
$$u = 0 \quad \text{“} \quad r = a,$$
$$u = f(r) \quad \text{“} \quad z = b.$$

Starting with the particular solution of (1),

$$u = \sinh(\mu z) J_0(\mu r), \tag{2}$$

and proceeding as in Art. 5, we get, if $\mu_1, \mu_2, \mu_3, \ldots$ are roots of

$$J_0(\mu a) = 0, \tag{3}$$

and
$$f(r) = A_1 J_0(\mu_1 r) + A_2 J_0(\mu_2 r) + A_3 J_0(\mu_3 r) + \ldots, \tag{4}$$

$$u = A_1 \frac{\sinh(\mu_1 z)}{\sinh(\mu_1 b)} J_0(\mu_1 r) + A_2 \frac{\sinh(\mu_2 z)}{\sinh(\mu_2 b)} J_0(\mu_2 r) + \ldots. \tag{5}$$

(*b*) If instead of keeping the convex surface of the cylinder at temperature zero we surround it by a jacket impervious to heat the equation of condition, $u = 0$ when $r = a$, will be replaced by $\frac{\partial u}{\partial r} = 0$ when $r = a$, or if $u = \sinh(\mu z) J_0(\mu r)$ by

$$\frac{dJ_0(\mu r)}{dr} = 0 \quad \text{when } r = a,$$

that is, by
$$-\mu J_1(\mu a) = 0$$

or
$$J_1(\mu a) = 0. \tag{6}$$

If now in (4) and (5) $\mu_1, \mu_2, \mu_3, \ldots$ are roots of (6), (5) will be the solution of our new problem.

(*c*) If instead of keeping the convex surface of the cylinder at the temperature zero we allow it to cool in air which is at the temperature zero, the condition $u = 0$ when $r = a$ will be replaced by $\frac{\partial u}{\partial r} + hu = 0$ when $r = a$, h being the coefficient of surface conductivity.

If $u = \sinh(\mu z)J_0(\mu r)$ this condition becomes

$$-\mu J_1(\mu r) + hJ_0(\mu r) = 0 \quad \text{when } r = a,$$

or
$$\mu a J_1(\mu a) - ahJ_0(\mu a) = 0. \tag{7}$$

If now in (4) and (5) $\mu_1, \mu_2, \mu_3, \ldots$ are roots of (7), (5) will be the solution of our present problem.

It can be shown that

$$J_0(x) = 0, \tag{8}$$

$$J_1(x) = 0, \tag{9}$$

and
$$xJ_1(x) - \lambda J_0(x) = 0 \tag{10}$$

have each an infinite number of real positive roots.* The earlier roots of these equations can be obtained without serious difficulty from the table for $J_0(x)$ and $J_1(x)$ at the end of this chapter.

ART. 21. DEVELOPMENT IN TERMS OF BESSEL'S FUNCTIONS.

We shall now obtain the developments called for in the last article.

Let $f(r) = A_1J_0(\mu_1 r) + A_2J_0(\mu_2 r) + A_3J_0(\mu_3 r) + \ldots$ (1)

μ_1, μ_2, μ_3, etc., being roots of $J_0(\mu a) = 0$, or of $J_1(\mu a) = 0$, or of
$$\mu a J_1(\mu a) - \lambda J_0(\mu a) = 0.$$

To determine any coefficient A_k multiply (1) by $rJ_0(\mu_k r)dr$ and integrate from zero to a. The first member will become

$$\int_0^a rf(r)J_0(\mu_k r)dr.$$

Every term of the second member will vanish by (9), Art. 19, except the term

$$A_k \int_0^a r\,[J_0(\mu_k r)]^2 dr.$$

$$\int_0^a r[J_0(\mu_k r)]^2 dr = \frac{1}{\mu_k^2}\int_0^{\mu_k a} x[J_0(x)]^2 dx = \frac{a^2}{2}\left([J_0(\mu_k a)]^2 + [J_1(\mu_k a)]^2\right)$$

by (6), Art. 19.

* See Riemann's Partielle Differentialgleichungen, § 97.

Hence $$A_k = \frac{2}{a^2([J_0(\mu_k a)]^2 + [J_1(\mu_k a)]^2)} \int_0^a r f(r) J_0(\mu_k r) dr. \quad (2)$$

The development (1) holds good from $r = 0$ to $r = a$ (see Arts. 6 and 15).

If μ_1, μ_2, μ_3, etc., are roots of $J_0(\mu a) = 0$, (2) reduces to

$$A_k = \frac{2}{a^2[J_1(\mu_k a)]^2} \int_0^a r f(r) J_0(\mu_k r) dr. \quad (3)$$

If μ_1, μ_2, μ_3, etc., are roots of $J_1(\mu a) = 0$, (2) reduces to

$$A_k = \frac{2}{a^2[J_0(\mu_k a)]^2} \int_0^a r f(r) J_0(\mu_k r) dr. \quad (4)$$

If μ_1, μ_2, μ_3, etc., are roots of $\mu a J_1(\mu a) - \lambda J_0(\mu a) = 0$, (2) reduces to

$$A_k = \frac{2\mu_k^2}{(\lambda^2 + \mu_k^2 a^2)[J_0(\mu_k a)]^2} \int_0^a r f(r) J_0(\mu_k r) dr. \quad (5)$$

For the important case where $f(r) = 1$

$$\int_0^a r f(r) J_0(\mu_k r) dr = \int_0^a r J_0(\mu_k r) dr = \frac{1}{\mu_k^2} \int_0^{\mu_k a} x J_0(x) dx = \frac{a}{\mu_k} J_1(\mu_k a) \quad (6)$$

by (5), Art. 19; and (3) reduces to

$$A_k = \frac{2}{\mu_k a J_1(\mu_k a)}; \quad (7)$$

(4) reduces to

$$A_k = 0, \quad (8)$$

except for $k = 1$, when $\mu_k = 0$, and we have

$$A_1 = 1; \quad (9)$$

(5) reduces to $$A_k = \frac{2\lambda}{(\lambda^2 + \mu_k^2 a^2) J_0(\mu_k a)}. \quad (10)$$

Prob. 28. A cylinder of radius one meter and altitude one meter has its upper surface kept at the temperature 100°, and its base and convex surface at the temperature 15°, until the stationary temperatures are established. Find the temperature at points on the axis 25, 50, and 75 centimeters from the base, and also at a point 25 centimeters from the base and 50 centimeters from the axis.

Ans. 29°.6; 47°.6; 71°.2; 25°.8

Prob. 29. An iron cylinder one meter long and 20 centimeters in diameter has its convex surface covered with a so-called non-conducting cement one centimeter thick. One end and the convex surface of the cylinder thus coated are kept at the temperature zero, the other end at the temperature of 100 degrees. Given that the conductivity of iron is 0.185 and of cement 0.000162 in C. G. S. units.

Find to the nearest tenth of a degree the temperature of the middle point of the axis, and of the points of the axis 20 centimeters from each end after the temperatures have ceased to change.

Find also the temperature of a point on the surface midway between the ends, and of points of the surface 20 centimeters from each end. Find the temperatures of the three points of the axis, supposing the coating a perfect non-conductor, and again, supposing the coating absent. Neglect the curvature of the coating. Ans. $15^\circ.4$; $40^\circ.85$; $72^\circ.8$; $15^\circ.3$; $40^\circ.7$; $72^\circ.5$; $0^\circ.0$; $0^\circ.0$; $1^\circ.3$.

Prob. 30. If the temperature at any point in an infinitely long cylinder of radius c is initially a function of the distance of the point from the axis, the temperature at any time must satisfy the equation $\frac{\partial u}{\partial t} = a^2\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right)$ (see Art. 1), since it is clearly independent of z and ϕ.

Show that

$$u = A_1 e^{-a^2\mu_1^2 t} J_0(\mu_1 r) + A_2 e^{-a^2\mu_2^2 t} J_0(\mu_2 r) \\ + A_3 e^{-a^2\mu_3^2 t} J_0(\mu_3 r) + \ldots,$$

where, if the surface of the cylinder is kept at the temperature zero, $\mu_1, \mu_2, \mu_3, \ldots$ are roots of $J_0(\mu c) = 0$ and A_k is the value given in (3) with c written in place of a; if the surface of the cylinder is adiabatic $\mu_1, \mu_2, \mu_3, \ldots$ are roots of $J_1(\mu c) = 0$ and A_k is obtained from (4); and if heat escapes at the surface into air at the temperature zero $\mu_1, \mu_2, \mu_3, \ldots$ are roots of $\mu c J_1(\mu c) - \lambda J_0(\mu c) = 0$, and A_k is obtained from (5).

Prob. 31. If the cylinder described in problem 29 is very long and is initially at the temperature 100° throughout, and the convex surface is kept at the temperature 0°, find the temperature of a point 5 centimeters from the axis 15 minutes after cooling has begun; first when the cylinder is coated, and second, when the coating is absent. Ans. $97^\circ.2$; $0^\circ.01$.

Prob. 32. A circular drumhead of radius a is initially slightly distorted into a given form which is a surface of revolution about the axis of the drum, and is then allowed to vibrate, and z is the ordinate of any point of the membrane at any time. Assuming that

z must satisfy the equation $\frac{\partial^2 z}{\partial t^2} = c^2\left(\frac{\partial^2 z}{\partial r^2} + \frac{1}{r}\frac{\partial z}{\partial r}\right)$, subject to the conditions $z = 0$ when $r = a$, $\frac{\partial z}{\partial t} = 0$ when $t = 0$, and $z = f(r)$ when $t = 0$, show that $z = A_1 J_0(\mu_1 r) \cos \mu_1 ct + A_2 J_0(\mu_2 r) \cos \mu_2 ct + \ldots$ where $\mu_1, \mu_2, \mu_3, \ldots$ are roots of $J_0(\mu a) = 0$ and A_k has the value given in (3).

Prob. 33. Show that if a drumhead be initially distorted as in problem 32 it will not in general give a musical note; that it may be initially distorted so as to give a musical note; that in this case the vibration will be a steady vibration; that the periods of the various musical notes that can be given are proportional to the roots of $J_0(x) = 0$, and that the possible nodal lines for such vibrations are concentric circles whose radii are proportional to the roots of $J_0(x) = 0$.

ART. 22. PROBLEMS IN BESSEL'S FUNCTIONS.

If in a problem on the stationary temperatures of a cylinder $u = 0$ when $z = 0$, $u = 0$ when $z = b$, and $u = f(z)$ when $r = a$, the problem is easily solved. If in (2), Art. 20, and in the corresponding solution $z = \cosh(\mu z) J_0(\mu r)$ we replace μ by μi, we can readily obtain $z = \sin(\mu z) J_0(\mu r i)$ and $z = \cos(\mu z) J_0(\mu r i)$ as particular solutions of (1), Art. 20; and

$$J_0(xi) = 1 + \frac{x^2}{2} + \frac{x^4}{2^2 . 4^2} + \frac{x^6}{2^2 . 4^2 . 6^2} + \ldots \tag{1}$$

and is real.

$$f(z) = \sum_{k=1}^{k=\infty} A_k \sin \frac{k\pi z}{b}$$

where
$$A_k = \frac{2}{b}\int_0^b f(z) \sin \frac{k\pi z}{b}\, dz \tag{2}$$

by Art. 9.

Hence
$$u = \sum_{k=1}^{k=\infty} A_k \sin \frac{k\pi z}{b} \frac{J_0\left(\frac{k\pi r i}{b}\right)}{J_0\left(\frac{k\pi a i}{b}\right)} \tag{3}$$

is the required solution.

A table giving the values of $J_0(xi)$ will be found at the end of this chapter.

Prob. 34. A cylinder two feet long and two feet in diameter has its bases kept at the temperature zero and its convex surface at 100 degrees Centigrade until the internal temperatures have ceased to change. Find the temperature of a point on the axis half way between the bases, and of a point six inches from the axis, half way between the bases. Ans. 72.°1; 80°.1.

ART. 23. BESSEL'S FUNCTIONS OF HIGHER ORDER.

If we are dealing with Laplace's Equation in Cylindrical Coordinates and the problem is not symmetrical about an axis, functions of the form

$$J_n(x) = \frac{x^n}{2^n \Gamma(n+1)}\left[1 - \frac{x^2}{2^2(n+1)} + \frac{x^4}{2^4 . 2!(n+1)(n+2)} - \ldots\right]$$

play very much the same part as that played by $J_0(x)$ in the preceding articles. They are known as Bessel's Functions of the nth order. In problems concerning hollow cylinders much more complicated functions enter, known as Bessel's Functions of the second kind.

For a very brief discussion of these functions the reader is referred to Byerly's Fourier's Series and Spherical Harmonics; for a much more complete treatment to Gray and Matthews' admirable treatise on Bessel's Functions.

ART. 24. LAMÉ'S FUNCTIONS.

Complicated problems in Potential and in allied subjects are usually handled by the aid of various forms of curvilinear coördinates, and each form has its appropriate Harmonic Functions, which are usually extremely complicated. For instance, Lamé's Functions or Ellipsoidal Harmonics are used when solutions of Laplace's Equation in Ellipsoidal coordinates are required; Toroidal Harmonics when solutions of Laplace's Equation in Toroidal coordinates are needed.

For a brief introduction to the theory of these functions see Byerly's Fourier's Series and Spherical Harmonics.

TABLE I. SURFACE ZONAL HARMONICS.

θ	$P_1(\cos\theta)$	$P_2(\cos\theta)$	$P_3(\cos\theta)$	$P_4(\cos\theta)$	$P_5(\cos\theta)$	$P_6(\cos\theta)$	$P_7(\cos\theta)$
0°	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
1	.9998	.9995	.9991	.9985	.9977	.9967	.9955
2	.9994	.9982	.9963	.9939	.9909	.9872	.9829
3	.9986	.9959	.9918	.9863	.9795	.9713	.9617
4	.9976	.9927	.9854	.9758	.9638	.9495	.9329
5	.9962	.9886	.9773	.9623	.9437	.9216	.8961
6	.9945	.9836	.9674	.9459	.9194	.8881	.8522
7	.9925	.9777	.9557	.9267	.8911	.8476	.7986
8	.9903	.9709	.9423	.9048	.8589	.8053	.7448
9	.9877	.9633	.9273	.8803	.8232	.7571	.6831
10	.9848	.9548	.9106	.8532	.7840	.7045	.6164
11	.9816	.9454	.8923	.8238	.7417	.6483	.5461
12	.9781	.9352	.8724	.7920	.6966	.5892	.4732
13	.9744	.9241	.8511	.7582	.6489	.5273	.3940
14	.9703	.9122	.8283	.7224	.5990	.4635	.3219
15	.9659	.8995	.8042	.6847	.5471	.3982	.2454
16	.9613	.8860	.7787	.6454	.4937	.3322	.1699
17	.9563	.8718	.7519	.6046	.4391	.2660	.0961
18	.9511	.8568	.7240	.5624	.3836	.2002	.0289
19	.9455	.8410	.6950	.5192	.3276	.1347	−.0443
20	.9397	.8245	.6649	.4750	.2715	.0719	−.1072
21	.9336	.8074	.6338	.4300	.2156	.0107	−.1662
22	.9272	.7895	.6019	.3845	.1602	−.0481	−.2201
23	.9205	.7710	.5692	.3386	.1057	−.1038	−.2681
24	.9135	.7518	.5357	.2926	.0525	−.1559	−.3095
25	.9063	.7321	.5016	.2465	.0009	−.2053	−.3463
26	.8988	.7117	.4670	.2007	−.0489	−.2478	−.3717
27	.8910	.6908	.4319	.1553	−.0964	−.2869	−.3921
28	.8829	.6694	.3964	.1105	−.1415	−.3211	−.4052
29	.8746	.6474	.3607	.0665	−.1839	−.3503	−.4114
30	.8660	.6250	.3248	.0234	−.2233	−.3740	−.4101
31	.8572	.6021	.2887	−.0185	−.2595	−.3924	−.4022
32	.8480	.5788	.2527	−.0591	−.2923	−.4052	−.3876
33	.8387	.5551	.2167	−.0982	−.3216	−.4126	−.3670
34	.8290	.5310	.1809	−.1357	−.3473	−.4148	−.3409
35	.8192	.5065	.1454	−.1714	−.3691	−.4115	−.3096
36	.8090	.4818	.1102	−.2052	−.3871	−.4031	−.2738
37	.7986	.4567	.0755	−.2370	−.4011	−.3898	−.2343
38	.7880	.4314	.0413	−.2666	−.4112	−.3719	−.1918
39	.7771	.4059	.0077	−.2940	−.4174	−.3497	−.1469
40	.7660	.3802	−.0252	−.3190	−.4197	−.3234	−.1003
41	.7547	.3544	−.0574	−.3416	−.4181	−.2938	−.0534
42	.7431	.3284	−.0887	−.3616	−.4128	−.2611	−.0065
43	.7314	.3023	−.1191	−.3791	−.4038	−.2255	.0398
44	.7193	.2762	−.1485	−.3940	−.3914	−.1878	.0846
45°	.7071	.2500	−.1768	−.4062	−.3757	−.1485	.1270

Table I. Surface Zonal Harmonics.

θ	$P_1(\cos\theta)$	$P_2\cos\theta$	$P_3(\cos\theta)$	$P_4(\cos\theta)$	$P_5(\cos\theta)$	$P_6(\cos\theta)$	$P_7(\cos\theta)$
45°	.7071	.2500	−.1768	−.4062	−.3757	−.1485	.1270
46	.6947	.2238	−.2040	−.4158	−.3568	−.1079	.1666
47	.6820	.1977	−.2300	−.4252	−.3350	−.0645	.2054
48	.6691	.1716	−.2547	−.4270	−.3105	−.0251	.2349
49	.6561	.1456	−.2781	−.4286	−.2836	.0161	.2627
50	.6428	.1198	−.3002	−.4275	−.2545	.0563	.2854
51	.6293	.0941	−.3209	−.4239	−.2235	.0954	.3031
52	.6157	.0686	−.3401	−.4178	−.1910	.1326	.3153
53	.6018	.0433	−.3578	−.4093	−.1571	.1677	.3221
54	.5878	.0182	−.3740	−.3984	−.1223	.2002	.3234
55	.5736	−.0065	−.3886	−.3852	−.0868	.2297	.3191
56	.5592	−.0310	−.4016	−.3698	−.0510	.2559	.3095
57	.5446	−.0551	−.4131	−.3524	−.0150	.2787	.2949
58	.5299	−.07[illegible]8	−.4229	−.3331	.0206	.2976	.2752
59	.5150	−.1021	−.4310	−.3119	.0557	.3125	.2511
60	.5000	−.1250	−.4375	−.2891	.0898	.3232	.2231
61	.4848	−.1474	−.4423	−.2647	.1229	.3298	.1916
62	.4695	−.1694	−.4455	−.2390	.1545	.3321	.1571
63	.4540	−.1908	−.4471	−.2121	.1844	.3302	.1203
64	.4384	−.2117	−.4470	−.1841	.2123	.3240	.0818
65	.4226	−.2321	−.4452	−.1552	.2381	.3138	.0422
66	.4067	−.2518	−.4419	−.1256	.2615	.2996	.0021
67	.3907	−.2710	−.4370	−.0955	.2824	.2819	−.0375
68	.3746	−.2896	−.4305	−.0650	.3005	.2605	−.0763
69	.3584	−.3074	−.4225	−.0344	.3158	.2361	−.1135
70	.3420	−.3245	−.4130	−.0038	.3281	.2089	−.1485
71	.3256	−.3410	−.4021	.0267	.3373	.1786	−.1811
72	.3090	−.3568	−.3898	.0568	.3434	.1472	−.2099
73	.2924	−.3718	−.3761	.0864	.3463	.1144	−.2347
74	.2756	−.3860	−.3611	.1153	.3461	.0795	−.2559
75	.2588	−.3995	−.3449	.1434	.3427	.0431	−.2730
76	.2419	−.4112	−.3275	.1705	.3362	.0076	−.2848
77	.2250	−.4241	−.3090	.1964	.3267	−.0284	−.2919
78	.2079	−.4352	−.2894	.2211	.3143	−.0644	−.2943
79	.1908	−.4454	−.2688	.2443	.2990	−.0989	−.2913
80	.1736	−.4548	−.2474	.2659	.2810	−.1321	−.2835
81	.1564	−.4633	−.2251	.2859	.2606	−.1635	−.2709
82	.1392	−.4709	−.2020	.3040	.2378	−.1926	−.2536
83	.1219	−.4777	−.1783	.3203	.2129	−.2193	−.2321
84	.1045	−.4836	−.1539	.3345	.1861	−.2431	−.2067
85	.0872	−.4886	−.1291	.3468	.1577	−.2638	−.1779
86	.0698	−.4927	−.1038	.3569	.1278	−.2811	−.1460
87	.0523	−.4959	−.0781	.3648	.0969	−.2947	−.1117
88	.0349	−.4982	−.0522	.3704	.0651	−.3045	−.0735
89	.0175	−.4995	−.0262	.3739	.0327	−.3105	−.0381
90°	.0000	−.5000	.0000	.3750	.0000	−.3125	.0000

TABLE II. BESSEL'S FUNCTIONS.

x	$J_0(x)$	$J_1(x)$	x	$J_0(x)$	$J_1(x)$	x	$J_0(x)$	$J_1(x)$
0.0	1.0000	0.0000	5.0	-.1776	-.3276	10.0	-.2459	.0435
0.1	.9975	.0499	5.1	-.1443	-.3371	10.1	-.2490	.0184
0.2	.9900	.0995	5.2	-.1103	-.3432	10.2	-.2496	.0066
0.3	.9776	.1483	5.3	-.0758	-.3460	10.3	-.2477	-.0313
0.4	.9604	.1960	5.4	-.0412	-.3453	10.4	-.2434	-.0555
0.5	.9385	.2423	5.5	-.0068	-.3414	10.5	-.2366	-.0789
0.6	.9120	.2867	5.6	.0270	-.3343	10.6	-.2276	-.1012
0.7	.8812	.3290	5.7	.0599	-.3241	10.7	-.2164	-.1224
0.8	.8463	.3688	5.8	.0917	-.3110	10.8	-.2032	-.1422
0.9	.8075	.4060	5.9	.1220	-.2951	10.9	-.1881	-.1604
1.0	.7652	.4401	6.0	.1506	-.2767	11.0	-.1712	-.1768
1.1	.7196	.4709	6.1	.1773	-.2559	11.1	-.1528	-.1913
1.2	.6711	.4983	6.2	.2017	-.2329	11.2	-.1330	-.2039
1.3	.6201	.5220	6.3	.2238	-.2081	11.3	-.1121	-.2143
1.4	.5669	.5419	6.4	.2433	-.1816	11.4	-.0902	-.2225
1.5	.5118	.5579	6.5	.2601	-.1538	11.5	-.0677	-.2284
1.6	.4554	5699	6.6	.2740	-.1250	11.6	-.0446	-.2320
1.7	.3980	.5778	6.7	.2851	-.0953	11.7	-.0213	-.2333
1.8	.3400	.5815	6.8	.2931	-.0652	11.8	.0020	-.2323
1.9	.2818	.5812	6.9	.2981	-.0349	11.9	.0250	-.2290
2.0	.2239	.5767	7.0	.3001	-.0047	12.0	.0477	-.2234
2.1	.1666	.5683	7.1	.2991	.0252	12.1	.0697	-.2157
2.2	.1104	.5560	7.2	.2951	.0543	12.2	.0908	-.2060
2.3	.0555	.5399	7.3	.2882	.0826	12.3	.1108	-.1943
2.4	.0025	.5202	7.4	.2786	.1096	12.4	.1296	-.1807
2.5	-.0484	.4971	7.5	.2663	.1352	12.5	.1469	-.1655
2.6	-.0968	.4708	7.6	.2516	.1592	12.6	.1626	-.1487
2.7	-.1424	.4416	7.7	.2346	.1813	12.7	.1766	-.1307
2.8	-.1850	.4097	7.8	.2154	.2014	12.8	.1887	-.1114
2.9	-.2243	.3754	7.9	.1944	.2192	12.9	.1988	-.0912
3.0	-.2601	.3391	8.0	.1717	.2346	13.0	.2069	-.0703
3.1	-.2921	.3009	8.1	.1475	.2476	13.1	.2129	-.0489
3.2	-.3202	.2613	8.2	.1222	.2580	13.2	.2167	-.0271
3.3	-.3443	.2207	8.3	.0960	.2657	13.3	.2183	-.0052
3.4	-.3643	.1792	8.4	.0692	.2708	13.4	.2177	.0166
3.5	-.3801	.1374	8.5	.0419	.2731	13.5	.2150	.0380
3.6	-.3918	.0955	8.6	.0146	.2728	13.6	.2101	.0590
3.7	-.3992	.0538	8.7	-.0125	.2697	13.7	.2032	.0791
3.8	-.4026	.0128	8.8	-.0392	.2641	13.8	.1943	.0984
3.9	-.4018	-.0272	8.9	-.0653	.2559	13.9	.1836	.1166
4.0	-.3972	-.0660	9.0	-.0903	.2453	14.0	.1711	.1334
4.1	-.3887	-.1033	9.1	-.1142	.2324	14.1	.1570	.1488
4.2	-.3766	-.1386	9.2	-.1367	.2174	14.2	.1414	.1626
4.3	-.3610	-.1719	9.3	-.1577	.2004	14.3	.1245	.1747
4.4	-.3423	-.2028	9.4	-.1768	.1816	14.4	.1065	.1850
4.5	-.3205	-.2311	9.5	-.1939	.1613	14.5	.0875	.1934
4.6	-.2961	-.2566	9.6	-.2090	.1395	14.6	.0679	.1999
4.7	-.2693	-.2791	9.7	-.2218	.1166	14.7	.0476	.2043
4.8	-.2404	-.2985	9.8	-.2323	.0928	14.8	.0271	.2066
4.9	-.2097	-.3147	9.9	-.2403	.0684	14.9	.0064	.2069
5.0	-.1776	-.3276	10.0	-.2459	.0435	15.0	-.0142	.2051

TABLE III.—ROOTS OF BESSEL'S FUNCTIONS.

n	x_n for $J_0(x_n)=0$	x_n for $J_1(x_n)=0$	n	x_n for $J_0(x_n)=0$	x_n for $J_1(x_n)=0$
1	2.4048	3 8317	6	18.0711	19.6159
2	5.5201	7.0156	7	21.2116	22.7601
3	8.6537	10.1735	8	24.3525	25 9037
4	11.7915	13.3237	9	27.4935	29.0468
5	14.9309	16.4706	10	30.6346	32.1897

TABLE IV.—VALUES OF $J_0(xi)$.

x	$J_0(xi)$	x	$J_0(xi)$	x	$J_0(xi)$
0.0	1.0000	2.0	2.2796	4.0	11.3019
0.1	1.0025	2.1	2.4463	4.1	12.3236
0.2	1.0100	2.2	2.6291	4.2	13.4425
0.3	1.0226	2.3	2.8296	4.3	14.6680
0 4	1.0404	2.4	3.0493	4.4	16.0104
0.5	1.0635	2.5	3.2898	4.5	17.4812
0.6	1.0920	2.6	3.5533	4.6	19.0926
0 7	1.1263	2.7	3.8417	4.7	20.8585
0.8	1.1665	2.8	4 1573	4.8	22.7937
0.9	1.2130	2.9	4.5027	4.9	24.9148
1.0	1.2661	3.0	4.8808	5.0	27.2399
1.1	1.3262	3.1	5.2945	5.1	29.7889
1.2	1.3937	3.2	5.7472	5.2	32.5836
1.3	1.4963	3.3	6.2426	5.3	35.6481
1 4	1.5534	3.4	6.7848	5.4	39.0088
1.5	1.6467	3.5	7.3782	5.5	42.6946
1.6	1.7500	3.6	8.0277	5.6	46.7376
1.7	1.8640	3.7	8.7386	5.7	51.1725
1.8	1.9896	3.8	9.5169	5.8	56.0381
1.9	2.1277	3.9	10.3690	5.9	61.3766

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